The constraint on public debt when $r < g$ but $g < m$

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Abstract

With real interest rates below the growth rate of the economy, but the marginal product of capital above it, the public debt can be lower than the present value of primary surpluses by the bubble premia on the debt. The government can potentially run a deficit forever. In a model that endogenizes the bubble premium as arising from the safety and liquidity of public debt, more government spending requires a larger bubble premium, but because people want to hold less debt, there is an upper limit on spending. Inflation reduces the fiscal space, financial repression increases it, and redistribution of wealth or income taxation affect the bubble premium in unconventional directions.

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1 Introduction

Almost every year in the past century (and maybe longer), the long-term interest rate on US government debt (r) was below the growth rate of output (g), and the gap between the two has increased in the last two decades. At the same time, the US data also strongly suggests that the marginal product of capital (m) has stayed relatively constant, well above the growth rate of output, so $g < m$. Panel (a) of Figure 1 shows expected long-run values of these three rates, while panel (b) instead uses geometric averages over the past 10 years. Panel (c) measures the marginal product of capital using capital income, as opposed to asset prices, which after subtracting for depreciation is higher than the growth rate of economy. Panel d) plots capital income and investment, noting that $g < m$ is typically stated as a condition for the dynamic efficiency of the economy (Abel et al., 1989), so the investment to capital ratio would be a lower bound for m.[1]

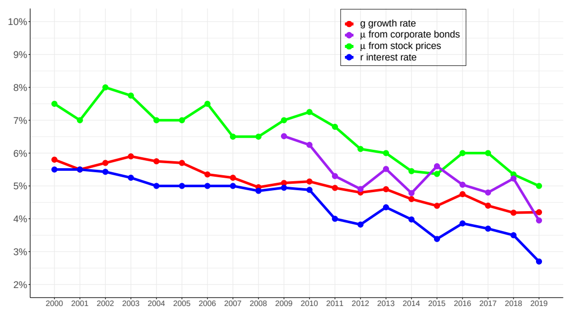
This paper investigates the implications for fiscal policy of having $r < g < m$. Section 2 goes through some simple, yet general, government budget constraint arithmetics to show that, in this case, the government can run a perpetual budget deficit. Yet, there is still a well-defined budget constraint. The debt now equals the present value of the ratio of primary surpluses to output discounted by $m - g$ plus the discounted bubble premia earned on the debt that equals $m - r$. It is not the gap $r - g$, but rather the gaps $m - g$ and $m - r$, what matters for public finances. These arithmetics open up several questions: why is $m > g > r$ and so what drives the two gaps? How does more government spending affect $m - g$ and $m - r$ in equilibrium? Is there a upper bound on the amount of spending for the bubble to be sustainable? How do monetary and fiscal policies affect the bubble premium, and through it do they tighten or loosen the government budget constraint?

Section 3 offers a model that answers these four questions by jointly determining r, g and m. Private investment is subject to idiosyncratic risk and to borrowing constraints. Public debt provides a safe haven from that risk, and an alternative store of value beyond the limits of private credit. These two properties are the most commonly estimated reasons for the $r - g$ differences that we observe in the data. Section 4 shows that in this model, higher public spending as a ratio of the debt raises the bubble premium $m - r$, but it lowers the amount of debt held by the public as a ratio of private assets. There is a maximal amount of public spending after which the bubble is not sustainable. This

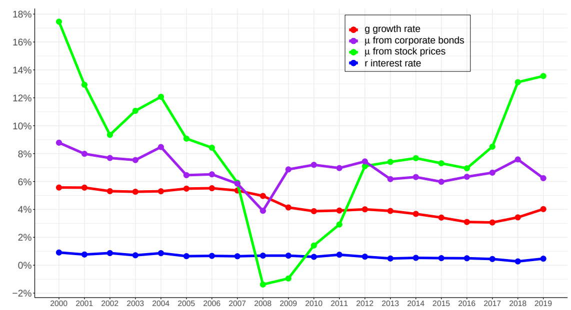
[1] For further discussions on the measurement of r, g, m, and for the international evidence, see Gomme, Ravikumar and Rupert (2011), Geerolf (2018), Barrett (2018), Mauro and Zhou (2020), Rachel and Summers (2019), Jordà et al. (2019).

Figure 1: The US marginal product of capital, growth rate, and interest rate since 2000

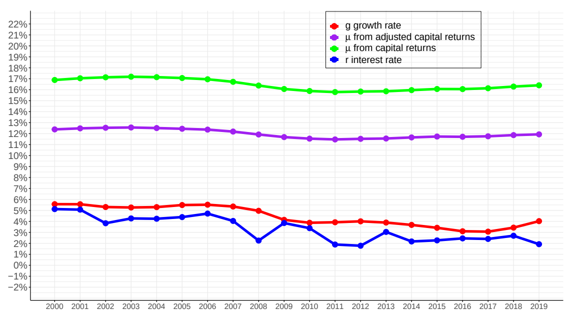
(a) Expected returns



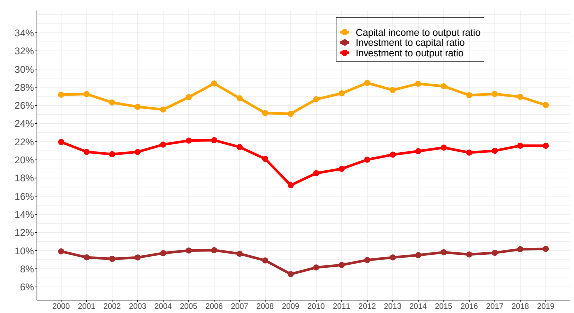
(b) Past average returns



(c) Capital returns



(d) Capital income and investment



Notes: Panel a) plots the expected returns on Treasury bonds, real GDP growth rate plus PCE inflation rate, stock returns, and returns on Baa corporate bonds, all at a 10-year horizon, according to the median respondent to the Survey of Professional Forecasters. Panel b) plots the geometric 10-year averages of Treasury 10-year bond returns, nominal GDP growth rate, r the returns on SP500 index, and returns on an index of Baa corporate bonds. Panel c) has the ay-year geometric average of a yield on a 10-year Treasury, the same output growth rate as in panel b), and a geometric 10-year average of the ratio of net value added minus labor expenditures to the corporate capital stock in the non-financial corporate sector from the Bureau of Economic Analysis' Survey of Current Business, and an adjusted return on capital that takes away 5% of GDP from capital income to account for land income, and 2/3 of proprietary income, attributed to a remuneration for labor. Panel d) plots point-in-time capital income series, now as a ratio of GDP, and the investment to capital and investment to output ratios using the BEA's data for non-financial corporate investment, capital stock and value added.

limit is tighter in economies that more financially developed, have less undiversifiable idiosyncratic risk, and less inequality. Section 5 considers various extensions of the model—aggregate risk, foreign demand for public bonds, a different fiscal rule for spending—and shows that the results are robust, but some new insights rise.

Section 6 shows that monetary and fiscal policies can have surprising effects on the fiscal space and capacity of the government through the bubble premium, once they affect r, g and m. Expected inflation is neutral, but inflation volatility lowers the safety of the public debt, so that ex ante it tightens the government budget constraint. There is no conflict in the mandates of the central bank and the fiscal authority, since delivering stable inflation is what creates the most fiscal space to raise public spending. Financial repression that coerces the private sector to hold government bonds at a below-market rate, creates fiscal space through an additional repression premium on the debt. However, it lowers growth because it worsens the allocation of capital. Perhaps more surprisingly, a tax-transfer systems that redistributes wealth to those that earn lower returns lowers the interest rate, keeping spending fixed, or lowers spending, keeping the interest rate fixed. It lowers the maximum spending before the bubble bursts. Therefore, there is a conflict between a fiscal authority that wants to spend more, and one that wants to redistribute more. Finally, a higher proportional income tax directly raises revenue, but indirectly reduces private credit. It shrinks the bubble in the public debt, even as it raises primary surpluses. In some cases, the effect o the bubble is larger, so that tax cuts can pay by themselves by raising economic activity and increasing the bubble premium on the debt.

All combined, the lesson is: in an economy that is dynamically efficient, but with a bubble in the public debt, there is still a constraint on how much the government can spend, and policies can shift this constraint in surprising ways through their effect on $m - g$ and $m - r$.

2 Debt arithmetics and the literature around it

Government budgets are not easy reads: borrowing comes through multiple instruments with different payment profiles and maturities, and spending and revenue lines depend on different bases and commitments. There are multiple r's and g's. Yet, some mild simplifications provide a clear statement of how debt will evolve over time. First, let s_t be the (net) public spending, or the (primary) public deficit. This includes both the flow of resources used by the government, as well as tax revenues and transfer spending. The question the paper asks is how large s_t can be. Therefore, for most of the paper, I assume it is exogenous.

Second, let the real market value of outstanding government debt be denoted by b_t. The return that private agents earn, (and the government pays) on this debt is r_t. This need not correspond to the promised yield on the debt, as there many be capital gains on long-term debt, or inflation affecting nominal debt.[2] The second assumption is that there is no aggregate uncertainty affecting either r_t or s_t. I will return to introducing uncertainty at different parts of the paper, showing it does not materially affect the main results.

The law of motion for the evolution of the public debt then is:

$$db_t = s_t dt + r_t b_t dt. \tag{1}$$

The debt increases because of spending plus paying interest on the debt (the total public deficit).

The third and final assumption is that output y_t grows at the rate g_t This is also deterministic. It may vary over time, but asymptotically it converges to a constant g. The key endogenous variables are then: b_t, r_t, g_t. Throughout, the empirically relevant case is when $r_t \leq g_t$ and $b_t \geq 0$. Further, I focus on a balanced growth path where the exogenous spending s_t and the endogenous debt asymptotically grow at the same rate as output, g.[3]

Working through this law of motion produces debt arithmetics that are, on the one hand, useful to understand the links between variables, but on the other hand, ultimately unsatisfactory, since they are identities and not constraints. This section shows that debt arithmetics answer some questions, yet raises just as many others.

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[2] To see this clearly, assume that there is a single nominal government bond, of which every instant a fraction ξ expires giving its holder a principal payment of 1, while the remaining $1-\xi$ pays no coupon but survives until next period. The expected maturity of government debt is $1/\xi$, matching the actual behavior of governments that perpetually roll over their debt, while keeping the maturity relatively stable. If $B_t \geq 0$ are the units outstanding of this bond, then its value in output units is: $b_t = B_t v_t / p_t$ where v_t is the nominal value (or price) of the bond, and p_t is the price level. Then, the return on the bond is: $r_t = \xi + (1-\xi)\frac{dv_t}{v_t} - \frac{dp_t}{p_t}$ where the first term is the coupon rate (or promised yield), the second term is the capital gain, and the third term is the inflation loss. Even if the government can choose the maturity ξ, or even how many bonds to sell B_t, the return on the government debt is endogenous as the market price adjusts as needed to clear markets.

[3] Could public debt grow at a faster rate than output? Since private consumption is bound by private gross income, then the savings of the households that hold the debt must grow to infinity. This is possible as long as they receive transfers from the government that rise over time as quickly as the debt. In turn this is consistent with the government budget constraint as long s_t/b_t converges to a constant equal to the growth rate of debt minus the real interest rate. While theoretically possible, an economy where the ratio of output to government spending goes to zero is not the most empirically interesting, so I focus on the balanced growth path instead.

2.1 Permanent deficits?

In the balance growth path, equation 1 implies that:

$$b = \max\left\{\frac{s}{g-r}, 0\right\}. \tag{2}$$

In spite of a fixed permanent deficit, with $r < g$, the government can still sustain positive debt.[4] From the opposite perspective, for a fixed amount of debt, the government can spend as a ratio of that debt the gap between the growth rate and the interest rate. Taking as given the value of b at the end of 2020 (127% of GDP), and a generous $r - g = 2\%$ this would imply that the permanent deficit could be 2.5% of GDP. But, surely a change in spending, would affect $r - g$ in equilibrium. The arithmetics show what is possible, and the tight link between each variable, but it is not enough by itself.

Perhaps there is no such equilibrium. If net spending is too high, the price of the debt will be zero, as the private sector refuses to hold this Ponzi scheme. Since b is the value of the debt, this corresponds to $b = 0$. Since the capital stock must be non-negative, perhaps this limit is reached when debt is equal to net private assets. If so, then this suggests an upper limit on spending between 4.8% and 7% of GDP, not much above the average primary deficit in 2010-20 (4.8%) or the Congressional Budget Office projection of a 4.6% deficit in 2050, suggesting that almost all of the fiscal profligacy from $r < g$ has already been used.[5] Yet, as spending changes this will likely affect the desirability of holding private and public assets in the economy. Net assets are not fixed, so while the debt arithmetics point to a limit, they can not go very far in telling when it will be hit.

2.2 Recurrent deficits?

Let the average interest rate between dates 0 and t be $\bar{r}_t = (\int_0^t r_s ds)/t$, and likewise for $\bar{g}_t$. A one -off spending splurge at date 0, Δs_0, raises debt-to-output t periods later by $\Delta b_t = e^{(\bar{r}_t - \bar{g}_t)t}\Delta s_0$. Even if the temporary increase in spending is very large, by pursuing

[4]The importance of $r < g$ may be best understood when $g = 0$. In this case, thinking of government debt as a consol, since $r < 0$, the debtholder is *paying* the government a fixed stream to hold the bond. This revenue is what finances the permanent spending.

[5]The lower bound comes from using the Bureau of Economic Analysis 2019 estimate that the capital stock was 2.1 times GDP, a net international investment position of -0.5 of GDP, and privately-held public debt plus debt of the Federal Reserve of 0.8 of GDP, for total assets of 2.4 times GDP. The upper bound comes from the Federal Reserve's Financial Accounts of the United States measure of private non-financial assets of 3.5 times of GDP.

this "deficit gamble" for enough years, there is a negligible decrease in spending needed to pay for the resulting debt in the distant future. However, the increase in spending is limited by output in that period. At the extreme, if the spending splurge was as high as output, with nothing left to consume, marginal utility of consumption would approach infinity, driving interest rate above the growth rate to infinity. Again, endogenizing r and g is crucial to understand even temporary gambles.

If a deficit gamble can be done once, why not frequently? Solving the debt dynamics in equation (1) forward to infinity:

$$b_0 = \lim_{T\to\infty} \left[-\int_0^T e^{-\bar{r}_t t} s_t dt + e^{-\bar{r}_T T} b_T \right]. \tag{3}$$

Since asymptotically $r < g$, then even if debt is never paid as it grows at the rate of output g, the limit of the second term in the right-hand side goes to infinity. This leads to the erroneous conclusion that any initial debt b_0 can be sustained, with no limit on public debt beyond the available resources in the economy, since deficit gambles can be repeated and rolled over in a Ponzi way. This is incorrect though. The limit of the sum is not the same as the sum of the limits. While the limit of the second term is plus infinity, the limit of the first term is minus infinity.

Rather, to solve the debt dynamics forward requires re-writing the flow budget constraint as $db_t - d_t b_t dt = s_t dt + (r_t - d_t) b_t dt$, for some discount rate $d_t > g_t$. Then, the limits are well defined and:

$$b_0 = -\int_0^\infty e^{-\bar{d}_t t} s_t dt + \int_0^\infty e^{-\bar{d}_t t} (d_t - r_t) b_t dt. \tag{4}$$

Even if $r < g$, there is still a mathematically well-defined limit on public debt or, equivalently, on how large can spending be. As a matter of arithmetics, a strictly higher sequence of d_t raises the first term towards zero and lowers the second one towards the initial value of the debt. But which is an appropriate d_t to use?

Let m_t be the marginal product of capital in the economy. This gives the private return of investing in production as opposed to in the government debt. Since, in the data $m_t > g_t$, this is an empirially legitimate choice for d_t. Moreover, it is a sensible choice since while equation 4 was a mathematical expression with no economic interpretation, the following equation:

$$b_0 = -\int_0^\infty e^{-\bar{m}_t t} s_t dt + \int_0^\infty e^{-\bar{m}_t t}(m_t - r_t) b_t dt \tag{5}$$

has an economic meaning as public debt is the sum of two terms.

The first term is the present value of spending, using the return on private assets as the valid stochastic discount factor, as one would for the payoffs in any other asset.[6] The marginal holder of the public debt could at the margin hold a unit of capital instead, so m_t is the relevant rate through which she would discount the holdings of the public debt. In turn, discounting by m_t is consistent with the transversality condition for those agents since optimal capital investment requires the marginal utility of consumption to grow at the rate of return on private assets. The condition $m > g$ for the integrals to be well defined is then just the dynamic efficiency condition.

The second term is the present value of the implicit government revenues due to it paying r_t in its government debt, when the marginal return in the private economy is instead m_t. The spread between the marginal product of capital in the economy and the interest rate paid on the debt measures how special debt is: its bubble premium, or convenience yield. Its product with the amount of outstanding debt is then the bubble revenue, or seignorage revenue. When $m_t > r_t$, the government can now pay for outstanding debt in part through these bubble revenues, so recurrent spending can be positive in present value.

In this expression, it is not $g_t - r_t$ that matters. Rather, $m_t - r_t$ is what drives the size of the bubble premium flows, while $m_t - g_t$ is what discounts future flows of spending and bubble premium. In an economy where $r_t \to g_t < m_t$ there is still a bubble premium, allowing for persistent government spending. In the neoclassical model, $m_t = r_t > g_t$ at all dates, the bubble is zero, and the conventional result follows that debt is equal to the present value of primary surpluses. It is only asymptotically that the discounting by $m - g$ and the bubble premia of $m - r$ combine to cancel out m and leave $g - r$ in equation (2).

[6] Recall that a valid stochastic discount factor (SDF) is one such that the expectations of its product with the market return is 1. Of course, the market return is itself a SDF.

2.3 Aggregate uncertainty?

Imagine now that spending, output, and the return on debt are all uncertain, so r_t is the *ex post* return on the debt. The debt constraint in equation 1 is unchanged. The stochastic discount factor D_t is also uncertain now. Letting $\mathbb{E}_t(.)$ be the expectations operator, integrating forward just as before gives:

$$b_0 = -\mathbb{E}_t\left(\int_0^\infty \frac{D_t s_t}{D_0} dt\right) + \mathbb{E}_t\left(\int_0^\infty \frac{D_t(m_t - r_t)b_t}{D_0} dt\right), \tag{6}$$

as long as the terminal condition $\lim_{t\to\infty} \mathbb{E}_t\left[(D_t/D_0)e^{\bar{g}_t t}(b_t/y_t)\right] = 0$ holds. This replaces the previous $d > g$ condition.

Debt is now equal to the expected present value of net spending plus the expected present value of bubble premium revenues. Again, choosing $D_t = e^{-\bar{m}_t t}$ is valid and economically meaningful if at the margin private agents can hold both the capital stock and public debt. The terminal condition arises from combining their transversality condition and the fact that the capital stock cannot be negative. Again also, a stochastic discount factor that is valid for asset pricing must price the capital stock, so the no arbitrage condition is $E_0\left[(D_t/D_0)e^{-\bar{m}_t t}\right] = 1$, and one obvious solution is $D_t = e^{-\bar{m}_t t}$. At the same time, just as with deterministic debt arithmetics, stochastic arithmetics open new questions, namely on how the discount factor endogenously co-moves with the bubble premium

2.4 Making progress

Debt arithmetics can provide useful insights. They show that there is a clear constraint on the public debt, that debt can exceed the present value of surpluses by the value of the bubble premium, and that recurrent and permanent spending are possible. Moving further though requires a model that endogenizes spending, the marginal product of capital, growth and interest rates, to make sense of how and when they vary with each other. The next section provides one such model that focuses on the safety and liquidity roles of government debt to generate its bubble premium.

The model builds on Reis (2013) and Aoki, Benigno and Kiyotaki (2010), by generating misallocation within a sector because more productive firms cannot borrow more than a fraction of their future revenue. Those papers studied the effect of large swings in capital flows from abroad, while this paper introduces uncertainty, and focuses on bubbles and public debt. For the most part, I assume a closed economy to further complement their analysis.

The focus on bubbles is shared with the production economies in Martin and Ventura (2012), Farhi and Tirole (2012), Aoki, Nakajima and Nikolov (2014), Hirano, Inaba and Yanagawa (2015) that is surveyed in Martin and Ventura (2018), but it is applied here to make sense of public debt and the intertemporal government budget constraint. Therefore, I do not study bubbles in private assets, which are covered there.[7]

There is an older literature on public debt in exchange economies with overlapping generations including economies with incomplete markets (Tirole, 1985, Santos and Woodford, 1997, Kocherlakota, 2008, Hellwig and Lorenzoni, 2009) with a focus on the link between $r < g$ and the existence and optimality of bubbles. In this paper, there is production so that there can be a marginal product of capital, and I focus on the fiscal implications of bubbles. To complement that literature, the model has agents that live forever, although most conclusions would still hold in a perpetual youth model.

Sims (2021) studies the interaction between distortionary taxes and a bubble term in the government budget constraint, but which he takes narrowly to refer solely to the seignorage due to printing money, so that the bubble premium is a nominal rate of return. That paper has no capital, so $m - g$ is just the nominal interest rate on government bonds, and the focus is on the inflation tax. Related, the focus on the sustainability of public debt when there is idiosyncratic risk and incomplete markets is shared with Bassetto and Cui (2018), Brunnermeier, Merkel and Sannikov (2020*b*), but they use this in a fiscal theory of inflation. This paper takes inflation as given, and I refer readers to these three papers for these complementary implications. Relative to all these papers, in this paper studies the interaction between fiscal capacity and other fiscal policies.[8]

Risk premia due to safety and liquidity are a major driver of the increasing wedge between m and r (Caballero, Farhi and Gourinchas, 2017, Farhi and Gourio, 2018, Mark, Mojon and Veldes, 2020, Negro et al., 2017, Ferreira and Shousha, 2020). Another part of it seems to be due to an increase in market power Farhi and Gourio (2018), Eggertsson, Robbins and Wold (2020). The work of Ball and Mankiw (2021) complements the one in this paper, by writing a model where instead market power generates the bubble premium on the debt, and likewise studying its implications for the constraints on public debt.

[7] I call the government revenue that results from $r < m$ a bubble premium, because of the link to this literature. The empirical literature that tries to measure it often calls it instead a convenience yield, and the theoretical literature that focuses on currency calls it seignorage.

[8] Ongoing work by Brunnermeier, Merkel and Sannikov (2020*a*) is closer to this paper by focussing on the safety of debt with idiosyncratic risk.

A different literature has focussed instead on the impact of aggregate uncertainty on the government budget constraint, but assuming a representative agent. It has shown that spikes in interest rates may make one-off deficit gambles fail (Ball, Elmendorf and Mankiw, 1998, Abel, 1992), that a stochastic discount factor provides the weights to consider different levels of spending Barro (2020), van Wijnbergen, Olijslager and de Vette (2020), and that there is a stationary distribution of debt-to-GDP that may include high levels Mehrotra and Sergeyev (2020). Given this complementary work, for most of the paper, I abstract from aggregate uncertainty to focus on idiosyncratic risk and on borrowing constraints leading to inequality and capital misallocation.

The two more direct intellectual antecedents of this paper are Blanchard (2019) and Jiang et al. (2019). Blanchard (2019) lays out arguments (and counter-arguments) for why, given $r < g$, governments can run prolonged deficits with minimal impact on fiscal space, or aim for a larger steady state debt-to-GDP.[9] This paper re-examines these conclusions when $g < m$, and investigates how fiscal, monetary, and financial polices affect the ability to undertake deficit gambles or carry larger debt.[10]

Jiang et al. (2019) argued that the stochastic discount factor that should be used in the government budget constraint is the same that should price risky assets in the economy, on which I built in the discussion above. They estimated that the present value of surpluses is quite small, so that the residual—the bubble term—must be very large. Since existing direct estimates of the convenience yield on the debt are an order of magnitude too low, they call this a "valuation puzzle".[11] This paper proceeds the investigation of the bubble premium, but does so theoretically, endogenizing it in a general-equilibrium model, and studying what forces generate it and what policies change it.[12] Future work can take on the next step of quantifying the effects discussed here towards solving the

[9] Blanchard and Weil (2001) also discuss the government budget constraint as a result of aggregate uncertainty leading to a Pareto inferior equilibrium. But, they do not discuss the bubble premium, do not have a borrowing constraint causing misallocation, and do not examine how more spending, other monetary and fiscal policies, affect the fiscal space.

[10] A slightly different perspective is that this paper reconciles Blanchard (2019) on what $r < g$ implies for public debt and spending, and Piketty (2013) on what $g < m$ implies for inequality and taxation. Both treated r, g, m as given, while this paper endogenizes them and discusses the interaction between taxation, spending, debt and inequality (see also citeMollRachelRestrepo.

[11] For estimates of convenience yields, see Negro et al. (2017), Jiang, Krishnamurthy and Lustig (2020), Rachel and Summers (2019)

[12] Jiang et al. (2020) also study policies in this context, but focussing on the covariation of s_t and m_t. Jiang et al. (2021) evaluate the implicit beliefs in the expectations operator of bondholders. Complementing this work focussed on aggregate uncertainty, in this paper I mostly assume a deterministic environment; I introduce aggregate uncertainty to show my conclusions are robust, but leave to these other papers the exploration of all their consequences.

puzzle they identify.[13]

3 A model where safety and liquidity are scarce

The model is a version of the neoclassical growth model where, besides the government, there is a representative firm, and many households that, because of incomplete markets, are unequal in opportunities and outcomes.

3.1 The firm

Because the focus is on public debt along a balanced growth path, I consider a simple linear economy, where there are no transition dynamics or aggregate risk, and all idiosyncratic risk is all iid time. A technology, which anyone can freely access, transforms quality-adjusted capital into output with a marginal product of capital of m_t.

In the population, there is a distribution $Q(q)$ of capital quality types $q \in [0,1]$, from which each household takes a draw at the start of each period. For each type, there is a continuum of households who get hit by a idiosyncratic depreciation shock to their capital $\delta(q)dz_t^{qi}$, which follow a Wiener process dz_t^{qi} such that $\int dz_t^{qi}di = 0$. The standard deviation of depreciation shocks $\delta(q) \geq 0$ weakly declines with quality. Each household's capital is therefore different in two ways: ex ante, through their type, and ex post through the realized depreciation. This is the only source of uncertainty and inequality in the economy: higher-quality types have better capital both in terms of their average value and in terms of their lower risk of wear-and-tear. There is positive mass of high-quality types, for whom $q = 1, \delta(1) = 0$ that can reap the full marginal product of capital at no risk, but there are also many others.

The neoclassical firm chooses how much of each capital to hire from each agent k_t^i by paying them r_t^{qi}:

$$\max \left\{ \int \int \left[m_t q_t dt - r_t^{qi} dt - \delta dz_t^{qi} \right] k_t^{qi} dQ(q) di \right\}. \tag{7}$$

[13] Two important considerations in the quantification that are strongly suggested by the results in this paper are that: (i) the correlation between idiosyncratic volatility and aggregate risk will amplify the bubble premium, and (ii) fiscal policies and the state of the business cycle affect both the bubble premium and the marginal product of capital, so their covariance can be substantial.

Therefore, for zero profits driven by competition, each quality type gets paid:

$$r_t^{qi} dt = m_t q_t dt - \delta dz_t^{qi}. \tag{8}$$

3.2 The households

Households live forever, discounting the future at rate $\rho > 0$ and obtaining utility from their individual consumption c_t^{qi}, and from the government services. I assume that the utility function is separable in these two sources of well-being so that, regardless of how important public services are, I can leave them out of the model as they have no effect in the equilibrium.

Household assets a_t^{qi} can be used to buy government bonds, b_t^{qi}, to invest in capital k_t^{qi}, or to lend to other households l_t^{qi}. The return on this last option is given by the interest rate r_t^l because there is a single private credit market. Households cannot short public debt, or invest negative amounts in capital, but they can borrow. However, they face a borrowing constraint in that the repayment of debt cannot exceed a fraction $\gamma < 1$ of the returns from capital investment in type q. As usual, this is justified by the borrower being able to abscond with all assets but for this share of the capital stock before it is time to pay the lender. Given the ex post depreciation risk, one can think of a mutual fund that pools capital across individuals within each quality type and borrows against it. Going forward, I refer to γ as the level of financial development of the economy, since the larger it is, the larger is the private debt market.

Combining all the ingredients, each household solves the following dynamic problem:

$$\begin{aligned}
&\max_{\left\{c_t^{qi}, b_t^{qi}, l_t^{qi}, k_t^{qi}\right\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t^{qi} dt\right] \\
&\text{subject to: } a_t^{qi} = b_t^{qi} + l_t^{qi} + k_t^{qi} \quad \text{with} \quad b_t^{qi} \geq 0, k_t^{qi} \geq 0 \\
&\qquad da_t^{qi} = (r_t b_t^{qi} + r_t^l l_t^{qi} + r_t^{qi} k_t^{qi} - c_t^{qi}) dt \\
&\qquad -r_t^l l_t^{qi} \leq \gamma m_t q_t k_t^{qi}
\end{aligned} \tag{9}$$

while taking initial assets a_0^i and the returns on investment as given.[14]

[14]Overlapping generations model can often be understood as models where some agents are constrained in how much they can borrow from the future. It is intriguing, but left for future research, to explore how those borrowing constraints would interact with the borrowing constraints in this paper.

3.3 Market clearing, equilibrium, and the first best

The economy is closed, so the market clearing conditions for the two assets are:

$$\int \int l_t^{qi} dQ(q)di = 0 \quad \text{and} \quad \int \int b_t^{qi} dQ(q)di = b_t. \tag{10}$$

Letting $k_t = \int \int k_t^{qi} dQ(q)di$ be the aggregate capital stock, I denote the ratio of government bonds to capital in equilibrium $\kappa_t = b_t/k_t$.

A *balanced-growth-path equilibrium* will then be an interest rate on government bonds r, a share of government bonds $\kappa_t \geq 0$, and a common growth rate for aggregate output, consumption, capital and public debt of g, given an exogenous choice of net spending as a ratio of the public debt s/b, such that: (i) the firm behaves competitively, so according to equation (8), (ii) the consumers behave optimally, so they solve the intertemporal problem in (9), (iii) the government debt satisfies the budget constraint in equation (2), and (iv) markets clear, as in the equations in (10). Importantly, I restrict attention to the case this paper wishes to study: when $r \leq g \leq m$ and there is permanent spending $s/b \geq 0$. Such an equilibrium may not exist. Indexing each equilibrium by the exogenous s/b, I will later show that it exists as long as $s/b \leq S$. This upper bound on spending, after which there is no equilibrium with a positive value of the public debt, is the *fiscal capacity* of the economy.[15]

The first best in the economy is simple. If $\gamma = 1$ then there are no credit frictions, so all households but the highest quality types prefer to lend to the $q = 1$ types, who invest in their superior capital stock. Therefore, we are in the textbook AK version of the neoclassical model, with $r = m > g(= r - \rho)$, contradicting the data. Since there is no bubble premium, the public debt must equal the present value of surpluses. Then, there is an equilibrium only if net spending is zero and so is the public debt, therefore the fiscal capacity is zero.

3.4 The roles of public debt

With incomplete financial markets, the most productive entrepreneurs cannot borrow as much as they would like to invest in their technology. Least productive and risky firms

[15] In this economy, κ could in principle be very large, arbitrarily so. In an overlapping generations model, there would be a limit to it, because the initial generation cannot save more than its income, and this would provide an additional constraint on spending.

are in business, creating a misallocation of resources that endogenously drives the wedge between the marginal product of capital and the growth rate of the economy.

Households are unable to trade the idiosyncratic depreciation risk that they bear if they invest in capital. The public debt is safe, since its returns are uncorrelated with the returns on individual capital. Public debt therefore provides *safety*, and will earn a corresponding premium.

Moreover, public debt provides an alternative to store wealth over periods for households if the constraint on private lending is too tight. Households are aware that they can become high-types in the futre, and want to store value for when this happens. Public debt therefore also provides outside *liquidity*, complementing the inside liquidity from private lending, and commanding a premium in return.

The two premia combined lead to $m > r$ creating a bubble premium. If the premia are large enough, it will also be that $g > r$ and the economy can sustain permanent net spending. The model therefore can generate the observed $r < g < m$ as a result of the desire for liquidity and safety in an unequal economy because of borrowing constraints. The misallocation of resources due to incomplete markets means that public debt is a bubble.

The model gives a simple vehicle to capture two important roles of public debt, and study whether persistent spending that tries to take advantage of the bubble premium is consistent with optimal behavior and markets clearing. If it is, then the size of this spending will endogenously determine r and g, as well as the fiscal capacity S. Policies and public debt will change the relative strengths of the safety and liquidity effects, and so can move the two key spreads, $m - r$ and $m - g$, as well as the fiscal capacity of the economy. In short, the model can give answers to the four questions that the debt arithmetics posed.

3.5 Simpler, and more complicated, versions

An even simpler version of the model provides much of the intuition. It assumes that there is no ex post uncertainty ($\delta(q) = 0$) and the ex ante heterogeneity is only over two types of agents:

$$q_t = \begin{cases} 1 & \text{if type } H, \quad \text{share } \alpha \\ 0 & \text{if type } L, \quad \text{share } 1 - \alpha \end{cases} \tag{11}$$

If the household is in the high group H, then quality is high (normalized to 1). By renting their capital they get the full marginal product of capital $r_t^H = m_t$. The remaining share of low-type households L have no access to production, as their capital is useless. The probability α plays an important role: the lower it is the fewer mass of agents have access to the good technology where all capital should be invested.

In this simple economy, there is no safety role of debt, since there is no uncertainty. All that remains is the liquidity role of public debt as a store of value, as all agents hope to be high-types in the future. The high types wish to borrow up to the debt limit and invest all in the productive capital, while the low types save in either government bonds or in private credit. For the economy to generate $r < g < m$, it must be that $\gamma < 1 - \alpha$: the borrowing constraint must be sufficiently tight that the economy cannot reach the first best, or, alternatively, there must be enough H-types wanting to lend through imperfect markets to the few E-types. The next section will cover the simple and more general models.

I will also consider more complicated versions that extend the results in the following section. First, one can have fiscal policy instead follow a rule that makes net spending as a ratio of private assets be an exogenous s/a. Second, one can include aggregate uncertainty by having the shocks dz_t^{qi} have an aggregate component: now, $\int dz_t^{qi} di = \zeta dz_t$ so ζ measures the correlation between aggregate and individual risk. Third, one can open the economy and have a foreign demand for public bonds according to a demand function $B(r)$ that weakly falls with the interest rate paid. Fourth, one can have diminishing returns to scale by writing the production function instead as $y_t = A_t \left(\int \int q_t k_t^{qi} dG(q) d(i) \right)^\theta$ and having TFP grow at an exogenous constant rate. Fifth, one can have persistence in the distribution of types q so that there is a wealth distribution. Sixth, one can bring in a different type of aggregate uncertainty, by letting the bubble on public debt stochastically pop, after which net public spending must be cut to zero. All of these bring interesting new considerations to the interplay between spending, the bubble premium, and fiscal capacity. Finally, section 6 introduces monetary and fiscal policies by further modifying the model.

4 Equilibrium gaps and fiscal capacity

I start by covering the simple two-type model, before moving to the general case.

4.1 The simple model

Since both bonds and private lending are safe investments, they must give the same return $r^l = r$. Then, if this interest rate is too low, the high-quality types would be able to borrow enough in private credit markets to reach the first best. For an equilibrium with $r < g < m$, in equilibrium it must be that $r > \gamma m$. At the same time, for any production to take place, it must be that in equilibrium $r < m$, otherwise no one would invest.

The appendix writes the dynamic problem solved by households. The high-quality types borrow as much as they can and invest it all in capital, not holding any government bonds. Their consumption and savings then grow at the rate:

$$\frac{\dot{a}_t^H}{a_t^H} = \frac{(1-\gamma)mr}{r-\gamma m} - \rho, \tag{12}$$

which exceeds $m - r - \rho$ because of their ability to leverage their investments. As for the L types, their capital is worthless so they split their assets between private lending and government bonds. The growth rate of the their assets and consumption is then:

$$\frac{\dot{a}_t^L}{a_t^L} = r - \rho. \tag{13}$$

Since each type is drawn from the same population, inequality is the difference between the two growth rates, which is $(m-r)/(r-\gamma m)$.

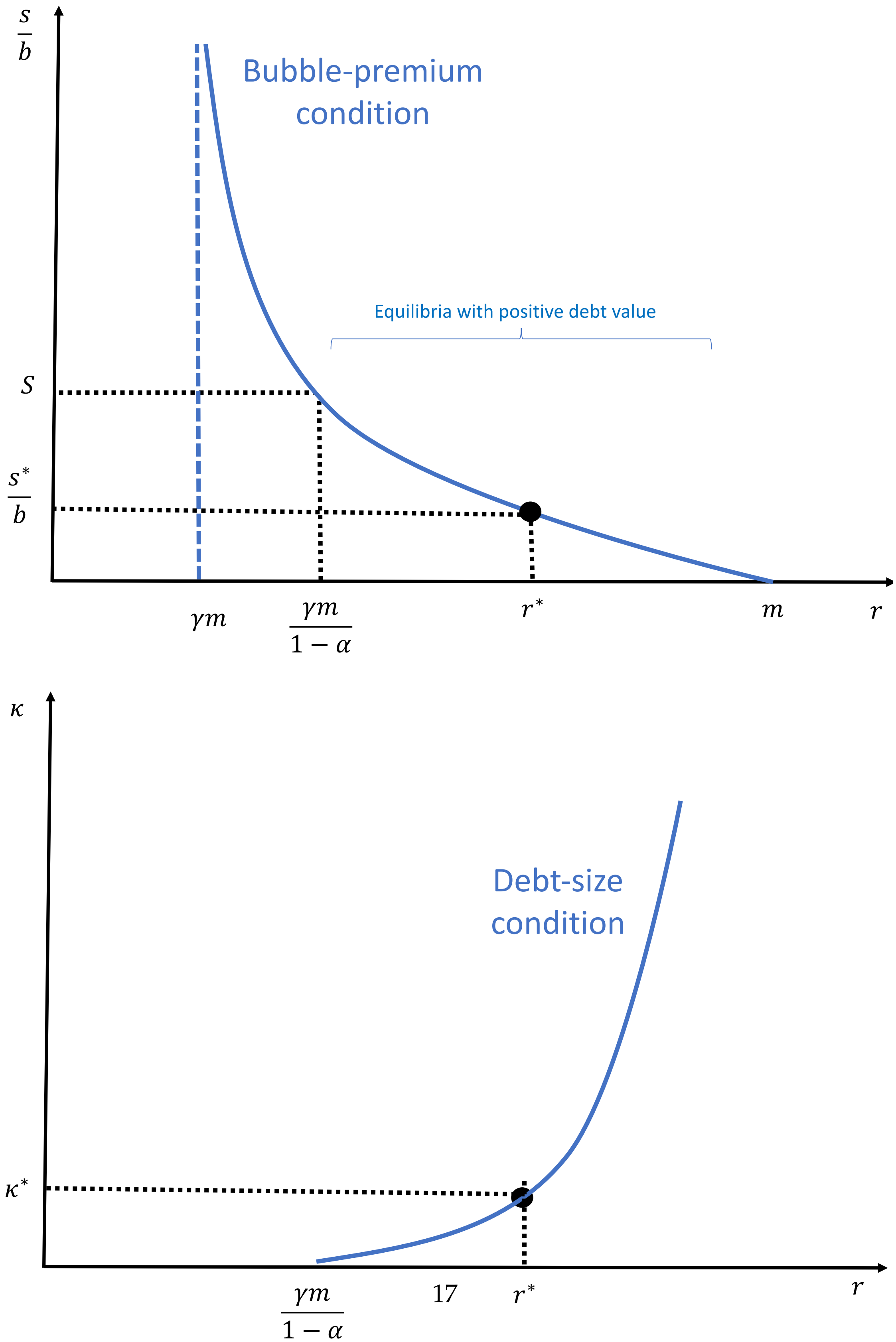
In a balanced-growth path, the growth rate of the economy is the weighted average of these two rates, with weights α and $1-\alpha$, respectively. In turn, the budget constraint of the government imposed that the growth rate is equal to $r + s/b$. Replacing out g, and rearranging gives the key equilibrium *bubble-premium condition*:

$$\frac{\alpha(m-r)}{1-\frac{\gamma m}{r}} = \rho + \frac{s}{b}. \tag{14}$$

Since the left-hand side is continuous and monotonic in r, this pins down the unique r solution of the model. The left panel of Figure 2 graphically represents this equilibrium.

At the same time, on aggregate $a = k + b$, so using the definition of the private capital to public debt ratio, market clearing in capital, and the fact that each type is an id draw: $\alpha a^E/k^E = 1 + \kappa$. The high-type assets equal $k^E - \gamma m k^E/r$, since they borrow to invest in

Figure 2: Equilibrium in the simple 2-type model



capital. The other equilibrium *debt-size condition* is:

$$\kappa = \frac{1-\alpha}{\alpha} - \frac{\gamma m}{\alpha r}. \tag{15}$$

This is depicted in the right panel of figure 2. It uniquely solves for the size of the public debt given a solution for r from the left panel.

There is only an equilibrium though if $\kappa \geq 0$. From the second equation, this is only the case when the interest rate is above $\gamma m/(1-\alpha)$, which from the first equation requires spending to not be too high. Combining all the results:

Proposition 1. *In the simple 2-type economy there is an equilibrium where government can run a permanent deficit paid for by the bubble premium and:*

- *More spending* (s/b) *requires a higher bubble premium* $m - r$.
- *More spending* (s/b) *lowers the ratio of public debt to private assets* κ.
- *More spending* (s/b) *increases inequality of consumption and asset growth.*
- *The fiscal capacity is:*

$$S = \frac{m}{\alpha}\left(1 - \frac{\gamma}{1-\alpha}\right) - \rho, \tag{16}$$

  so it is smaller if the marginal product of capital is lower (low m*), the economy is more financially developed (high* γ*), or if there are more high productivity types (high* α*)*

Some of these conclusions may seem surprising. But they follow naturally if the specialness of public debt arises from it allowing people to store their liquidity. When the government spends more, the bubble premium must be higher to sustain the extra permanent spending. But this requires the bond holders, who are the poorer and less productive households, to earn lower returns at the same time as the equity holders, who are richer, can leverage up more and earn higher returns. Therefore, inequality rises. At the same time, the higher spending comes with lower bond holdings, as the households prefer to lend in the private credit market instead. If the spending increase is too high, then the bubble pops, and there is no equilibrium with a positive value for debt since that much spending requires such low interest rates that no one wants to lend to the government. The fiscal capacity depends on the desirability of public bonds relative to private credit. If the economy is financially developed or has many investment opportunities, the bubble premium is lower because the private economy is able to allocate resources better and provide higher returns in credit markets. There is less room to finance public spending.[16]

4.2 The general economy

In the general economy, it is still the case that $r^l = r$, and it must be that $r > \gamma m$ otherwise the economy would reach the first best.

Starting with the household problem, because each type is ex ante identical, her choices of consumption and investment are going to be the same. As the appendix shows, optimal consumption requires that $c_t^{qi} = \rho a_t^{qi}$. Rearranging the budget constraint for each agent in the economy, assets grow according to: $da^{qi} = [r - \rho + (mq - r)(k/a)^q]\,dt - \delta(q)(k^q/a^q)a_t^{qi}dz_t^{qi}$. By market clearing, the growth rate in the balanced growth path is given by:

$$gdt = \int\int \left(\frac{da^{qi}}{a^{qi}}\right)\left(\frac{a^{qi}}{a}\right) dG(q)di. \tag{17}$$

Now, the iid assumption implies that each type has the same assets at the start of the period. Moreover, from the government budget constraint, the growth rate must equal $r + s/b$. Combining all of these into the previous equation, gives:

$$\rho + \frac{s}{b} = \int (mq - r)\left(\frac{k}{a}\right)^q dG(q). \tag{18}$$

Depending on their quality, different types of agents sort into different classes according to their investment decisions. For those, whose q is lower than r/m, productivity is too low. They prefer to invest zero and invest all their assets in either private credit markets or in the public debt. The next class is made of those with quality above r/m but below a threshold q^*. Those invest in capital according to its Sharpe ratio: $(k/a)^q = (mq - r)/\delta(q)^2$. For those with lower quality, this is less than their assets, so they invest the remainder in the public bonds or lending. For those with higher quality, they start borrowing in private credit markets, but their borrowing constraint is still slack. Finally, those with with $q > q^*$, do not buy any public bonds, borrow up to the limit and, and invest everything in their superior capital: $(k/a)^q = r/(r - \gamma mq)$. The appendix shows that a sufficient condition for $q^* > r/m$ is that there is a $q > 0$ such that

[16] The graphs would suggest that $r \in [\gamma m/(1-\alpha), m]$ and that $\kappa \in [0, (1-\gamma)/\alpha]$. However, it must be that $s/b \geq 0$ or that $g \geq r$. The appendix shows that this puts an upper bound on the interest rate $\hat{r} < m$, and therefore an upper bound on κ as well that is below $(1-\gamma)/\alpha$.

$r\delta(q)^2 > 0$. That $q^{<}1$ is guaranteed by the fact that there is a positive mass of agents with $q = 1, \delta(1) = 0$. Combining these investment choices with the previous equation gives the general model's equilibrium *bubble-premium condition*:

$$\rho + \frac{s}{b} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)} \right)^2 dQ(q) + \int_{q^*}^{1} \frac{(mq - r}{1 - \frac{\gamma mq}{r}} dQ(q). \tag{19}$$

As in the simple economy, the market clearing conditions imply that: $1/(1+\kappa) = k/a$. But, aggregating over the different types: $k/a = \int \int (k/a)^q (a_t^{qi}/a) dG(q) di$. Combining these two equations with the investment choices discussed above gives the second equilibrium *debt-size condition* in the general model:

$$\frac{1}{1+\kappa} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)^2} \right) dQ(q) + \int_{q^*}^{1} \frac{r}{r - \gamma mq} dQ(q). \tag{20}$$

The equilibrium is the joint solution for r, κ over the two equations as a function of the exogenous s/b. From here it follows that:

Proposition 2. *In the general economy there is an equilibrium where government can run a permanent deficit paid for by the bubble premium and:*

- *More spending (s/b) requires a higher bubble premium $m - r$.*
- *More spending (s/b) lowers the ratio of public debt to private capital κ.*
- *More spending (s/b) increases inequality of consumption and asset growth between those at the top of the income distribution (with $q > q^*$) and those at the bottom (with $q < r/m$).*
- *There is a finite fiscal capacity S, which is smaller if the marginal product of capital is lower (low m), the economy is more financially developed (high γ), if there are more high productivity types (lower $G(q^*)$), or if there is less idiosyncratic risk in the economy (weakly lower $\delta(q)$ for all q)*

All of these properties mirror those in the simple model, with one addition: the consideration of idiosyncratic risk. In the simple model, there were only high and low quality types. In the general model, there is also an intermediate type, which finds refuge in the public debt because it provides some safety against the risk of capital investment. The bubble premium now includes also a safety premium because of the demand for public debt from these agents.

As the proposition shows, both the liquidity and the safety premium work in the same direction. More spending still requires a higher bubble premium because of the debt arithmetics. This still comes with less public debt relative to private capital, as the public debt is less attractive, and it still hurts the bottom of the income distribution because they disproportionately hold the public debt. There is still a finite fiscal capacity, which is smaller if private credit markets work better.

The novelty is that less idiosyncratic risk now directly reduces the demand for safety. Therefore, it lowers the safety premium, and so it reduces fiscal capacity.[17]

5 Other considerations

This section considers extensions of the model covering many of the other considerations considered in the literature.

5.1 Net spending as a ratio of private assets

If policy chose an exogenous amount for s, then spending would become an irrelevant fraction of income as time goes by. Spending must grow at the rate at which debt, capital, or output grow. Therefore the policy choice is just at what level to set s_0. In the model, this choice was made relative to the public debt at that period, since this followed naturally from the debt arithmetics in section 2. At the same time, the propositions shows that when s/b rises, then b/k falls, leaving open whether spending was actually higher or not.

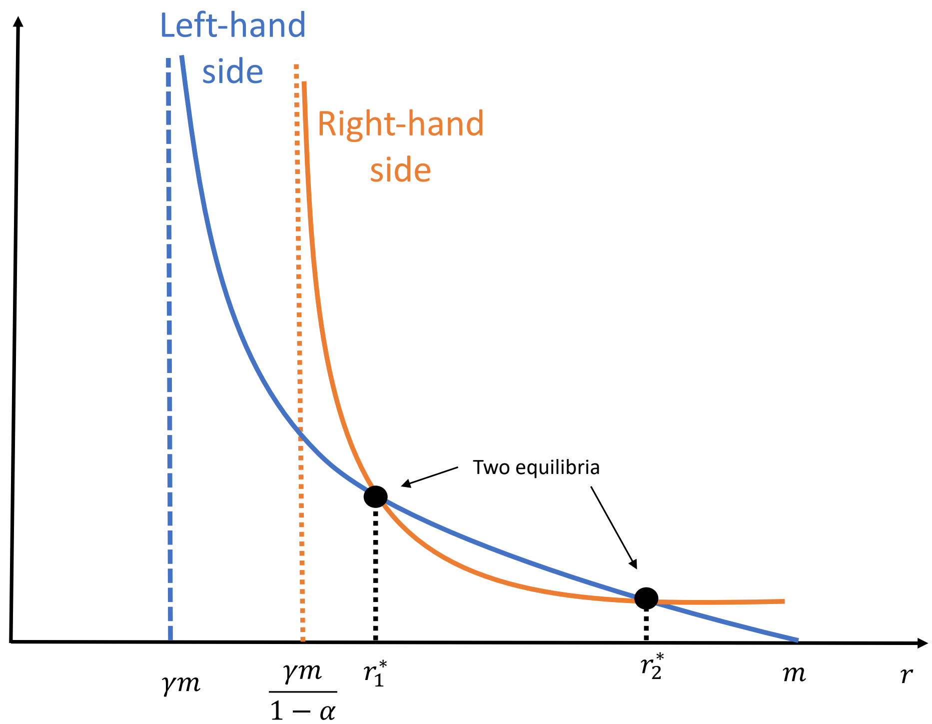
Since initial assets are also exogenous, a natural alternative is to have spending set as a ratio of assets. So, now s/a is exogenous. In the simple model, the two equilibrium conditions in equations (14)-(15) now have to be solved simultaneously for the interest rate and bond holdings. Combining the two, in terms of the new exogenous variable, the equilibrium interest rate now is:

$$\frac{\alpha(m-r)}{1-\frac{\gamma m}{r}} = \rho + \frac{s}{a}\left(1 - \frac{\alpha}{1-\gamma m/r}\right)^{-1}. \tag{21}$$

As before, the left-hand side fall monotonically with r starting at infinity when r is close to γm, and falling to 0 when $r = A$. Now, the right hand side also fals with r, starting

[17] For readers interested in isolating the safety premium, the appendix solves the case where there is a single type q, so there is no liquidity premium in the debt.

Figure 3: Equilibrium with a fiscal rule for s/a



at infinity when r is close to $\gamma m/(1-\alpha)$, the lowest level it can be that is consistent with non-negative bond holdings, and falls to ρ. This is displayed in figure 3.

Clearly there are two cases. If s/a is too high, then there is no intersection between the two curves. In other words, there is still finite fiscal capacity S. However, now with low s/a, there are two possible equilibria. In one of them, interest rates are low, the bubble premium is high, but bond holdings are small. On the other, interest rates are high, so the bubble premium is low, but bond holdings are high. In the two, the implicit bubble revenues for the government are the same, since they finance the same amount of spending.

In the low interest rate equilibrium, the same implications as in proposition 1 hold. But now, it is also possible that a change in spending (or even a sunspot) leads to a sudden "run on the debt", where interest rates spike, and he value of bond-holdings crashes.

5.2 Aggregate uncertainty

Next, consider the case where the shocks that hit the economy have an aggregate component, since: $dz_t^{qi} = \zeta dz_t + d\hat{z}_t^{qi}$. The shocks dz_t hits all, so ζ is the covariance of shocks across agents, while the shock $d\hat{z}_t^{qi}$ is idiosyncratic and integrates to zero across households.

Of the two equations determining equilibrium, the debt-size condition in equation (20), is clearly unchanged since it was not affected by the shocks. The bubble-premium condition in equation (19), is now different because it has a new term in the right-hand side:

$$\rho + \frac{s}{b} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)(1+\zeta)} \right)^2 dQ(q) + \int_{q^*}^{1} \frac{(mq - r}{1 - \frac{\gamma mq}{r}} dQ(q)$$
$$- \left[\int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)(1+\zeta)} \right) dQ(q) + \int_{q^*}^{1} \frac{\delta(q)(mq - r}{1 - \frac{\gamma mq}{r}} dQ(q) \right] \zeta dz_t. \tag{22}$$

First, an aggregate shock that raises depreciation now lowers the right-hand side. This lowers interest rates, just as raising spending didm as the economy has less output now. Second, for there to be a BGP now, s/b cannot be constant. Rather, spending would have to fall whenever there is a bad depreciation shock; by how much is given by the expression in square brackets. Conditional on doing so, then a change in average spending as a ratio of bonds would have the exact same effects as described in proposition 5.3. Third, because the aggregate shocks raise uncertainty, the first term on the right hand side is also now lower: agents want to hold less capital. This raises the safety premium on the debt, and so lower interest rates and raises fiscal capacity.

5.3 Foreign demand for public bonds

Imagine now that, on top of the demand from households with lower-quality capital, there is also a foreign demand for domestic government bonds. Therefore total government bonds b_t, also include an amount $B(r - r^f)$, growing with the economy at rate g, so it does not become negligent. This amount falls with the gap between the domestic interest rate and a foreign counterpart r^f, as the returns offered to foreigners are smaller. Of the two equilibrium equations of the model, only the debt-size condition in equation

(20) changes, as the left-hand side has a new term:

$$\frac{1}{1+\kappa} + \frac{B(r)}{a_0} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)^2} \right) dQ(q) + \int_{q^*}^{1} \left(\frac{r}{r - \gamma m q} \right) dQ(q). \tag{23}$$

At one extreme, imagine that there is perfect capital mobility, so unless $r = r^f$ there is infinite demand or short selling by the foreigners of the public debt. Given the fixed r at its international level, then the other equilibrium bubble-premium condition, equation (19), shows that there is a unique s/b consistent with that equilibrium. The reason is that for a given r, the bubble premium $m - r$ is now exogenous in this economy. The government budget constraint imposes that s/b can no longer be freely chosen, but there is a unique level of it consistent with a positive value of government debt. The fiscal capacity S is now equal to this level. For the fixed interest rate, more financial development (γ), for instance, still raise the fiscal capacity as before.

At the other extreme, imagine that $B(r - r^f)$ is inelastic, or a constant. Then, nothing changes in the analysis of the model. There is a constant in the left-hand side of equation (23), and if foreigners demand more bonds, this obviously raises the share of public debt to private capital in the economy, and raises the fiscal capacity S.

In between these two cases, if $B(r - r^f)$ has a negative finite derivative, the model works in the same way and proposition still holds. Now, more spending, lowers domestic interest rates, which lowers demand for public bonds from domestic households and now also from foreigners. Therefore the bubble revenue increases by less, and the fiscal capacity is reached faster.

6 Monetary-fiscal policy trade-offs

Redistributive, fiscal, monetary, and financial regulation policies affect the equilibrium growth rate and interest rate in the economy, interacting with the amount of government spending. Therefore, they affect the bubble premium on the government debt, and so the ability to run perpetual deficits and their size. Insofar as the policies are chosen by a different policymaker than the one choosing public spending, conflicts will arise. This section studies these effects, and the trade-offs they give rise to.

I investigate this in two complementary ways. First, I ask whether the policy lowers or raises the fiscal capacity S. I answer this by changing the policy, while keeping spending s/b fixed at a level that is below this capacity both before and after the policy. Second, I instead ask whether, letting government spending responds to a change in policy to keep the interest rate fixed, does the policy lead permanent spending s/b to rise or fall. In this second question, by keeping r fixed, the bubble premium $m - r$ is unchanged with the policy, so the effect of the policies on spending will come from whether they increase the demand for government bonds. If s/b is higher after the policy, I say that *fiscal space* has increased.[18]

6.1 Monetary policy: inflating the debt or deflating the bubble?

Assume that inflation is positive and stochastic:

$$dp_t/p_t = \pi_t dt + \sigma_\pi dz_t^\pi, \tag{24}$$

where π_t is the expected inflation rate, and dz_t^π are aggregate shocks to inflation, uncorrelated with the idiosyncratic shocks to the depreciation of the capital stock.[19] I take these as given, implicitly assuming that the classical dichotomy holds. It would be standard to assume there is a central bank that chooses a nominal interest rate according to a Taylor rule, in which case π_t could be its inflation target and dz_t^π map into monetary shocks. Introducing nominal rigidities, and studying how they affect the not just inflation but also the equilibrium in the model is left for future work.

If $\sigma_\pi = 0$, then nothing of substance changes in the analysis. In fact, this was already allowed for in the notation by calling r_t the real return on the bonds. A higher π_t simple leads the nominal price of the bond to rise faster over time leaving r_t unchanged.[20]

Shocks to inflation make a difference. The debt dynamics are now given by:

$$db_t = s_t dt + r_t b_t dt - b_t \sigma_\pi dz_t^\pi, \tag{25}$$

since r_t is an ex ante real return, but ex post an inflation shocks lowers the real value of the debt. A large positive shock lowers the real value of the debt, which would loosen the

[18]One potential use of this second question would be in applying the ideas in this paper in a fiscal theory of the price level. The adjustment in s, a real variable, could come about by an adjustment in the price level.

[19]An unfortunate assumption is that inflation is iid, when in fact in the data, it is quite persistent. This can significantly distort inferences on how much inflation can inflate away the debt (Hilscher, Raviv and Reis, 2014) Therefore, the variance σ_π^2 must be thought of as significantly larger than just the measured variance in the data.

[20]Using the case of the bond as a nominal annuity introduced earlier, now the nominal price of the bonds v_t is simply given by $\dot{v}_t/v_t = (r_t - \xi + \pi)/(1 - \xi)$, but the real value b_t is unchanged.

government's budget constraint and directly allow for s_t to rise that period. At the same time, in the other direction, unexpectedly lower inflation raises the real value of the debt. On average these ex post effects cancel out. Ex ante though, inflation uncertainty matters. Assume that the fiscal rule is $(s_t/b_t)dt = \sigma dt + \sigma_\pi dz_t^\pi$ neutralizing the ex post effect so the economy is still in a deterministic balanced growth path. Then, all that remains is the uncertainty from investing in the government bonds. All else equal, this uncertainty lowers bond holdings, as public debt is less safe. The interest rate rises, and this shrinks the fiscal capacity. Also, adjusting σ to keep r fixed, then this lowers bondholdings κ. Therefore, the implicit revenues from the bubble are lowers, and permanent spending has to fall.

Collecting all of these:

Proposition 3. *Changes in expected inflation π_t have no consequences on the spending ability of the government. A higher variance of inflation σ_π lowers the fiscal capacity S and the fiscal space s/b.*

During times of fiscal trouble, discussions about inflation tend to focus on the ex post benefits from inflating the debt. However, if the spending was being financed using the bubble premium that results from $r < g < m$, then the desire to inflate it away leads in equilibrium to an inflation risk premium in bonds. This partly offsets the safety and store of value premium on those bonds. Therefore, the bubble is smaller. As a result, attempts to inflate away the debt when bond holders are forward-looking reduce the bubble value of the debt, and so tighten the budget constraint of the government.

To loosen the debt burden on the fiscal authority, the best action for monetary policy in this economy would be to stabilize inflation as much as possible. This has a footprint on the government's budget, because it permanently lowers the inflation risk premia that must be paid on the debt, creating fiscal capacity. Price stability generates fiscal resources, while a switch to monetary instability can trigger a rise r and cause a sovereign debt crisis.

6.2 Financial repression: sacrificing private credit

A common form of financial repression is to force the financial system to hold under-priced government bonds. This is sometimes done by central banks that force banks to hold required reserves and do not pay interest on them. Other times, it is done by financial regulators that require financial institutions' assets to be held in safe investments for macro-prudential reasons, when in many countries the only safe asset is a liability from the government. In more extreme times, of war or after large expenses, the government may legally or through strongly-stated moral suasion force financial markets to lend funds to support public programs at a fixed discounted rate.

In the model, separate the public debt into a coerced and a voluntary amount:

$$b_t = b_t^c + b_t^v. \tag{26}$$

The voluntary debt is freely chosen by agents given a return r_t just as before. The coerced debt is mandatory and pays a below-market return. For simplicity, I assuem that every agent holds the same amount of coerced debt, and that the forced return is zero. The debt dynamics are therefore now given by:

$$db_t = s_t dt + r_t b_t^v dt. \tag{27}$$

The household choices on consumption and investment do not change with respect to the their voluntarily-disposed assets $a_t - b_t^c$. The bubble-premium condition in equation 19 that determines the interest rate is now:

$$\rho + \frac{s}{b} - r\left(\frac{b^c}{b}\right) = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)}\right)^2 dQ(q) + \int_{q^*}^{1} \left(\frac{mq - r}{1 - \frac{\gamma mq}{r}}\right) dQ(q). \tag{28}$$

The only difference is the new term on the left-hand side. It implies that the left-hand side now also falls with r, just as the right-hand does.

All else equal, an increase in the share of debt that is coerced will raise interest rates. The implicit revenue from financial repression lowers the need to have the bubble premium to finance the spending. At the same time, the voluntary bond-holding can be smaller. Both effects combined mean that the fiscal capacity S rises. Alternatively, keeping the bubble premium fixed, if r is unchanged, the right-hand side of the equation above is unchanged. Then, a higher b^c/b must allow for a higher s/b and the fiscal space rises. Collecting these results:

Proposition 4. *An increase in the share of public debt that is coerced raises the fiscal capacity S and the fiscal space s/b.*

Financial repression imposes a repression premium on the coerced debt. This unambiguously loosens the government budget constraint, which perhaps explains why this policy is so often used when governments get in fiscal trouble. At the same time, it also lowers growth because it worsens the allocation of capital. If tax revenues depended on economic activity, the decline caused by repression would automatically raise net spending. Moreover, repression affects the chances that there are financial crises, and these can require very large increases in spending s.[21]

6.3 Redistributive policy: lowering inequality versus raising public spending

Households with access to high-quality capital earn higher returns and so have higher income than those who are unfortunate to have low-quality capital. Since all households are ex ante identical, a utilitarian social planner would be tempted to address this inequality by taxing the former and redistributing to the latter. This may even raise welfare by providing social insurance.

Usually, redistribution comes with distortions to incentives. Since these will be studied in the next case, here I consider a best-case scenario where the redistribution is done through a tax-and-transfer on the initial assets of households. It taxes a share ψ of the assets of the high-quality types (those with $q > q^*$) and transfers its proceeds directly to the lower types that only earn the safe rate r on their income (those with $q < r/m$). The debt dynamics are unchanged since the program does not generate net revenues for the government. However, now the key equilibrium condition that determines interest rates becomes:

$$\rho + \frac{s}{b} = \int_{r/m}^{q^*} \left(\frac{mq - r}{\delta(q)} \right)^2 dQ(q) + (1 - \psi) \int_{q^*}^{1} \left(\frac{mq - r}{1 - \frac{\gamma mq}{r}} \right) dQ(q). \tag{29}$$

An increase in ψ lowers the left-hand side. It therefore raises the equilibrium interest rate on government debt and lowers the public bonds as a ratio of private capital that is held in equilibrium. This brings the economy closer to the point where agents to not want to hold debt. Therefore, it lowers the fiscal capacity S. In turn, it follows immediately from the equation above that, keeping r fixed, a higher ψ lowers the right-hand side, and so leads to a fall in fiscal space s/b.

Proposition 5. *A larger redistribution program (ψ) lowers the fiscal capacity S and the fiscal space s/b.*

[21]See Reis (2020), Acharya, Rajan and Shim (2020) for the interaction of these fiscal footprints of macro-prudential policy.

The intuition is that more redistribution implies that the low-quality types have more assets that they want to lend out as they await for a good entrepreneurial opportunity. The financial friction prevents this lending to go to higher-quality capital, trapping assets with their poor owner, as those who want to borrow cannot do so as much as they have fewer assets. The growth rate of the economy falls, shrinking the ability to sustain government spending. Proposition 5.3 already showed that more government spending increases inequality. Conversely, reducing inequality through redistribution lowers the spending the government can do.

This gives rise to an interesting policy trade-off. A policymaker that is focused on inequality and approves a large transfer program will constrain the ability of a different policymaker that is focused instead on spending, say to provide public services or infrastructure. A conflict will arise between the two. In a political system where parties alternate in power and have different preferences for inequality vis-a-vis public spending. It is well known, empirically and theoretically, that polarization of this type may lead to over-spending. The result above notes that because it also lowers fiscal space and fiscal capacity, political polarization heightens the risk that the public debt bubble pops.

6.4 Fiscal policy: tax cuts that pay for themselves?

Consider now the effects of distortionary taxation. Namely, all income is taxed at the rate τ. This not only lowers the returns on investment, but it also tightens the borrowing constraint because the entrepreneurs cannot pledge the tax bill for repayment of their debts. The budget constraint of the household in equation (9) therefore changes to:

$$\begin{aligned} da_t^{qi} &= (1-\tau)(r_t b_t^{qi} + r_t^l l_t^{qi} + r_t^{qi} k_t^{qi} - c_t^{qi})dt, \\ -r_t^l l_t^{qi} &\leq (1-\tau)\gamma m_t q_t k_t^{qi}. \end{aligned} \tag{30}$$

These tax revenues will now subtract from the net spending of the government.

The solution of this model is a little more involved, so it is in the appendix, which shows:

Proposition 6. *A marginal increase in the tax rate starting from zero (τ) may raise or lower fiscal capacity S but it lowers fiscal space s/b.*

Taxing income raises fiscal revenues. All else equal, the direct effect of this is to increase the fiscal capacity, since the government can use the new revenues to spend more.

As usual this policy also lowers the returns to investment, income and tax revenues; the well-known disincentive effects of taxes. More interesting, in the model, the tax lowers the ability of high-quality types to borrow. This directly lowers private investment and growth, and with it the bubble premium. Around zero taxes, it turns out that this second effect dominates, so the increase in tax revenues is more than offset by the fall in bubble revenues. Therefore, the fiscal space falls. As for the fiscal capacity, the results are ambiguous because the tax was akin to a fall in development of the financial market (a lower γ) which by itself raises capacity (proposition 5.3). Whether this effect, or the extra revenues, raise or lower S will depend.[22]

The question of whether tax cuts ever pay for themselves is a classic one in economics. The empirical debate revolves around measuring whether the tax base rises after a fall in the tax rate, because it incentivizes investment or labor supply. The perspective offered in the above lesson is quite different. First, because it suggests that deficit-financed tax cuts, by increasing the public debt, raise a source of revenue for the government. Second, because it suggests measuring how $m - r$ and $m - g$ responds to a tax cut. This would combine estimates of multipliers with estimates of direct crowding-in effects of tax changes on interest rates, but where the elasticity of investment to interest rates is irrelevant. These are intriguing cues for future research to pursue.

7 Conclusion

Public debt is expected by 2021 to exceed 120% of GDP on average across the advanced economies, matching or exceeding the previous peak in the last 140 years, which had been hit in 1945. At the same time, interest rates relative to the growth rate of the economy are low in most advanced economies, even relative to a history where $r < g$ quite frequently.[23] This paper asked what is the constraint on public debt when there is a bubble ($r < g$) but the economy is dynamically efficient ($g < m$). Quite differently from previous results in dynamically inefficient economies, it found that there was still a well-defined government budget constraint. It is relaxed by a bubble premium on the debt, which is the difference between the return that private agents can earn on the marginal unit of capital as opposed to lending to the government. This makes permanent deficits feasible even if the public debt is positive, but their size may be quantitatively small, and

[22]Distortionary taxes will lower the growth rate but, in this second-best economy with incomplete markets, they may raise some measures of welfare.

[23]Sources: IMF Fiscal Monitor of October 2020, and Mauro and Zhou (2020).

spending more will affect the bubble premium changing debt arithmetics. Monetary and fiscal policies may unexpectedly bring the existing public debt closer (or further away) from the fiscal capacity limit. In sum, thinking through the ability to spend allowed for by $r < g < m$ led to lessons on the limits and dangers of permanent spending.

Working through debt arithmetics led to four questions, stated in the introduction. First, why is $r < g < m$? The paper provided a model that matches the empirical measures that public debt provides safety and liquidity creating a bubble premium / convenience yield / seignorage that can pay for some spending. Second, how does spending affect the equilibrium bubble premium and bond holdings, as well as other variables? The model predicts that more spending raises the bubble premium but lowers bond-holdings, increasing inequality along the way. Third, is the fiscal capacity of the economy—the largest spending feasible without driving the value of debt to zero—finite, and what does it depend on? There is a finite limit to spending, and it is smaller if the economy is less productive, has more developed private financial markets, and less idiosyncratic risk. How do policies affect fiscal capacity and fiscal space? Inflation backfires, repression works but is costly, redistribution results in a tighter government budget constraint, and distortionary income taxation changes the size of the public debt bubble b y changing the allocation of capital, on top of its usual revenue and Laffer-curve effects.

Some of the results were surprising, while others perhaps less so, but all together they lay out clear policy trade-offs. Also, they leave out some new well-define empirical challenges: How much does the bubble premium respond to shocks to government spending? How much does it change with other policies? How do different compositions of the public debt affect the overall bubble premium? Can the three-way interaction between the stochastic discount factor, the bubble premium, and the amount of debt make quantitative progress on solving the debt valuation puzzle? Do the considerations of $r < g < m$ quantitatively change the strategic incentive for countries to default well before they reach their fiscal capacity? What is clear for now is that there are no fiscal free lunches and yet economists may have to re-think the advice they give to countries to avoid a fiscal crisis.

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Appendix – For Online Publication

This appendix contains further derivations not in the main manuscript

A The household problem, and q^*

The household problem in (9) can be written as (dropping superscripts):

$$\max_{\{c_t/a_t, k_t/a_t\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t dt \right]$$

$$\text{subject to:} \quad da_t = [r_t + (m_t q_t - r_t)(k_t/a_t) - c_t/a_t]\, a_t dt - \delta(k_t/a_t) a_t dz_t \tag{A1}$$

$$0 \leq k_t/a_t \leq \frac{r_t}{r_t - \gamma m_t q_t}$$

For generality, let the conditional distribution across types be $Q(q'|q)$. Then, the state variables for the household are her assets-type pair. Ignoring the constraints on capital, the associated Bellman equation is:

$$\rho V(a,q) = \max\{ \log(c) + V'(a,q) \left[r + (mq - r)\left(\frac{k}{a}\right) - \frac{c}{a} \right] a + \tag{A2}$$

$$\left(\frac{V''(a,q)}{2}\right)\left(\frac{k}{a}\right) \delta^2 a^2 + \int (V(a,q') - V(a,q)) dQ(q'|q) \} \tag{A3}$$

where $V'(.) \equiv \partial V(.)/\partial a$. It is standard to derive that at an optimum:

$$c = \rho a \tag{A4}$$

$$\frac{k}{a} = \frac{mq - r}{\delta(q)^2} \tag{A5}$$

$$V'(.) = \frac{1}{\rho a} \tag{A6}$$

$$\lim_{t \to \infty} e^{-\rho t} V'(.) a_t = 0 \tag{A7}$$

Combining the third and fourth equation, it is clear that the transversality condition is always respected as long as $\rho > 0$.

Recall that $\delta(q)$ decreases monotonically with q. Therefore, the optimal k/a above is monotonically increasing in q. It then follows that imposing the constraints leads to those with $q < r/m$ choosing $k/a = 0$, while those with a high q will be at the borrowing constraint and so have $k/a = r/(r - \gamma m q)$. This finishes the solution to the household problem.

Next comes showing that $r/m < q^* < 1$, recalling that q^* is defined by:

$$\frac{mq^* - r}{\delta(q^*)^2} = \frac{r}{r - \gamma m q^*} \Leftrightarrow \tag{A8}$$

$$(\gamma m^2 q^{*2} - (1+\gamma) mrq + r(r + \delta(q^*)) = 0. \tag{A9}$$

Start with the case where $\delta(q^*)$ is constant, so this is a quadratic equation of the form $f(q^*) = 0$ with a single root in $[0, 1]$. Next, it is easy to see that $f(0) = r\delta(0)$ and $f'(0) = -(1+\gamma)mr$. Therefore, as long as $r\delta(0)^2 > 0$ we will have $q^* > 0$. Next, note that $f(1) = (m-r)(\gamma m - r) < 0$ since there is a positive mass of agents with $q = 1, \delta(1) = 0$. Therefore, $q^* < 1$

B An upper bound on the interest rate

Since $s/b = g - r$ then the condition $g \geq r$ reduces to $s/b \geq 0$. Using equation (14), this condition becomes:

$$r^2 - (m - \rho)r - \gamma m \rho \geq 0 \tag{A10}$$

It is easy to show that the quadratic has a single root in $[\gamma m/(1-\alpha), m]$, call it $\hat{r}$, and that this root is interior, proving that there is an upper bound on an upper bound on the interest rate $\hat{r} < m$. Since κ increases with r, this also puts an upper bound on κ that is below $(1-\gamma)/\alpha$.

C An economy with only a safety, but no liquidity, premium

In this case, there is a single type, say $q = \eta$, and no private credit is possible, so $\gamma = 0$. In this case, there is no role for the public debt in allowing low-types to transfer value into the future when they might become high-types. Only the role for public debt as providing safety remains, so this special case allows us to isolate and study it.

In this case, the growth rate of assets is

$$\frac{\dot{a}_t}{a_t} = r - \rho + \left(\frac{m\eta - r}{\delta}\right)^2. \tag{A11}$$

Since the left-hand side is equal to g, which in turn from the government budget constraint is equal to $s/b + r$, then the equilibrium interest rate solves

$$r = \eta m - \sigma\sqrt{\frac{s}{b} + \rho} \tag{A12}$$

Next, since all households choose to hold the same share of assets in bonds, the equilibrium bonds held as a ratio of assets is:

$$\frac{b}{a} = 1 - \frac{m\eta - r}{\delta^2}. \tag{A13}$$

Recalling that $1 - b/a = 1/(1+\kappa)$ and using the solution for r form above this becomes:

$$\kappa = \frac{\delta}{\sqrt{\frac{s}{b} + \rho}} - 1 \tag{A14}$$

Note right away that an increase in uncertainty (δ) raises the desire for precautionary savings. So, it raises the holdings of government debt κ, while pushing down the interest rate r.

An increase in permanent spending as a ratio of debt lowers both r and β fall, just as happened in the simple model with only a demand for liquidity, and in the general model with a demand for safety and liquidity. The intuition is that more spending requires a larger bubble premium. But, since the bubble premium is solely a safety premium, it must be that households are investing more in the risky technology. Because this increases overall risk, then the safety of government debt is more valuable, its bubble premium rises, and so more persistent spending is possible. Higher spending also raises inequality since the lower safe interest rates induce households to invest more in their risky technologies, which have dispersed ex post returns. Finally, the upper bound on spending so that the government can spend forever is now:

$$S = \delta^2 - \rho. \tag{A15}$$

There has to be enough risk in the economy to drive r sufficiently down and create a bubble premium in the public debt. As in the general model, if the economy becomes less risky with financial development, then the upper bound S falls

In short, the two motives for why $r < g$ in the model— the uses of public debt as a store of value and as a safe harbor—complement each other.