EMERGENCE AND EVOLUTION OF COOPERATION FOR SURVIVAL: A CONTINUOUS TIME MODEL

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ABSTRACT. Considering a large homogeneous population, where individuals rely on the availability of a resource for their survival, we introduce a continuous time model for the availability of the resource in time and for each individual. Cooperation consists in transfers of the resource between members in a given group, from those having a surplus towards those having a shortage of the resource. The model explores essential questions regarding the importance of cooperation: what are the characteristics of populations where cooperation is valuable, versus individualist populations, where cooperation destroys value; how large should be the groups of cooperating entities, so that expected lifetimes increase? For the answering the latter question, in a first step, we characterise the optimal cooperation, which maximises the expected lifetime of generic, randomly selected entities in the population. Secondly, we explore the same question when self-interested entities choose strategically wether to cooperate or not, and we define the stable cooperative groups in terms of Nash equilibria. This allows to understand when and to what extent strategic cooperation leads to inefficiencies as compared with optimal cooperation. We express these inefficiencies both in terms of group size difference and of differential in the corresponding expected lifetime.

1. INTRODUCTION

We consider a large homogeneous population, where individuals rely on the availability of a resource for their survival, and introduce a continuous time model for the availability of the resource in time and for each individual. In this context, cooperation is defined as transfers of the resource between members in a given group, from those having a surplus towards those having a shortage of the resource. As cooperation creates dependencies between members, the evolution of the resource availability has to be considered jointly, at the level of a group. By relying on the theory of Markov processes, we effectively measure the effects of cooperation, as increasing expected lifetimes of the members in such cooperating groups. This allows to characterise the level of optimal cooperation, that leads to a maximal expected lifetime for the whole population. Achieving such an optimal cooperation would a priori require the presence of a regulator or central planner that forces groups of a certain size to form. The question that arises is whether or not the population will reach by itself the optimal level of cooperation without such an intervention of a central planner.

To tackle this important question, we derive endogenously the size of cooperating groups, using a continuous time stochastic game, where groups may decide to enlarge and include new individuals, with whom to cooperate from that point in time on, and also exterior individuals decide on wether to join or not such groups. Once formed, a group follows the rules of cooperation without defection. Here, the underlying assumption is that members are linked by a binding agreement. This

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assumption allows to focus on the process by which groups expand. Isolated individuals (exterior to a group) and the cooperating groups are assumed to be self-interested and aiming at maximising their expected lifetime. Assuming that the game evolves as Nash equilibria are played, we define the stable coalitions, as those groups that will never enlarge, because either they are not attractive to be entered by new members, or because they reject new members. Actions are based on the observation of the state process summarising the need of the resource at the level of an existing group. We account for adverse selection by considering that the information about the state process of isolated individuals is private to the individuals.

We then analyse the conditions under which cooperation emerges and how it evolves through time, leading either to the existence of stable coalitions (finite by nature) or infinite size coalitions. We also analyse when and to what extent strategic cooperation leads to inefficiencies as compared with optimal cooperation, that would be imposed by a central planner. We express these inefficiencies both in terms of group size difference and of loss of expected lifetime, for wide ranges of the parameters, using a numerical approach.

The individuals in our model (that will be called entities) can be economic agents, households, production and supply units, financial institutions etc. By the intended generality of its formulation, the model we propose may apply in such diverse fields as biology, political economy, finance, management, economic and social exchange, whenever one can abstract from some specific phenomena such as social norms or kin selection in human populations. In its essence, our work builds on the concept of reciprocal altruism, introduced by Trivers [23] in biology, who showed that even a selfish individual will come to the aid of an unrelated other, provided there is a sufficiently high probability the aid will be repaid in the future. Such a concept was vastly used in the economic literature since.

In our approach, cooperation emerges because of an uneven distribution of resources through time and space, as at a same point in time, some entities have a surplus of the resource, while others a shortage, and an entity may find itself in any of these roles within its lifetime. Cooperation therefore is nothing but a mean of managing the risks related to the availability of the resource in the future, based upon the idea of reciprocity. In this respect, our model is connected to some body of literature on risk sharing or mutual insurance in populations, with the difference that we measure value in terms of expected lifetime and not using utility functions. Building on initial work of Bala and Goyal [2] modelling network formation as a non-cooperative game, several papers study risk-sharing networks. Kocherlakota [14], Kimball [13], Coate and Ravallion [8] or, more recently Ligon, Thomas, and Worrall [16] proposed economic models and show that risk sharing among non-altruists may occur through repeated interaction in repeated game models with self-interested agents, even without binding contracts. Generosity today is justified by expected future reciprocity. In large homogeneous populations, theoretical results suggest that the larger the population, the higher is the per capita utility from risk sharing, see for example Genicot and Ray [7]. This implies that a Pareto optimal solution to risk sharing for risk averse agents would be to form as large a group as possible.

Our work results in a more contrasted picture, as we show that even when expected lifetimes are to be maximised by a central planner, individualistic populations do exist, where entities are better off without any form of cooperation, or sometimes small groups represent the optimal outcomes. These results are not justified by the existence of frictions, but simply by the incorporation into a continuous time setting of a richer and more realistic dynamics of the population's level of resources through time. Indeed individuals possess a rate of surplus creation and depletion and an average time of survival in absence of a surplus. These elements are key to understand the long term dynamics of the resource availability with versus in absence of cooperation. Depending on these parameters, we show that in the limit of large groups, infinite cooperation leads either to population decline and eventual extinction, or at the opposite, resource accumulation leading to abondance and infinite lifetimes for everyone.

Therefore, our theoretical results are in line with numerous field evidence showing that there are occurrences where smaller groups do better than larger groups with respect to risk sharing. For example, Deaton [3] (section 5.3) or research papers such as Townsend [22], Udry [24], Grimard [9], Fafchamps and Lund [5], Murgai et al [19], Morduch [18] test for full consumption insurance at the village (community) level in developing countries or ancient populations. All of these papers reject complete risk-sharing at the level of the community and find evidence of only partial cooperation.

The paper is organised as follows. In Section 2 we propose a simple Markov chain model for describing the resource need of an entity without cooperation. We then proceed to introduce cooperation and describe how it affects the evolution of the individual needs of the members of the group. In Section 3, we compute explicitly the expected lifetime of a coalition of size N and derive the mathematical properties of the optimal expected lifetime. In terms of ranges of only three parameters we can identify the populations that satisfy an optimal infinite cooperation, versus a finite one. We then turn in Section 4 to the modelling of the self-organisation process of entities in cooperating groups, using a continuous time strategic game and the notion of stable coalitions. We then analyse or not this process reaches by itself an optimal level of cooperation, without the intervention of a central planner. This topic is investigated both theoretically and using a numerical approach in Section 5.

2. The model

We introduce a continuous time model to study the emergence of cooperation within a large, homogeneous population composed of entities that can be economic agents, households, production and supply units etc.. The population relies on the availability of a certain resource for its survival. There is a state process for each entity that indicates the severity of its need of resource at a point in time, with three possible states: surplus, neutral and distress.

The entities evolve independently one from the other, unless they decide to cooperate and form a group. We shall call such groups a *coalition*, as the group forms specifically for the purpose of cooperating. Another reason for using this term, is that in the context of the strategic game in Section 4, coalitions will be considered to act together, as one unit, relative to the rest of the players. When an entity is not part of a coalition and it reaches the state of distress, nothing more happens and we can consider it as extinct. In other words, distress is an absorbing state for an entity in absence of cooperation. Inside a coalition on the other hand, each entity is linked to the others by relationships of borrowing and lending of the resource. These are aimed at improving the lifetime of the coalition's members, with entities in a surplus state ceding the surplus to the distressed entities. If all members of a coalition are distressed, entities cannot borrow the resource today or in the future and such a coalition is considered extinct.

To sum up, cooperation consists in relations of borrowing and lending the resource. It is not a priory clear that such a behaviour of cooperation is beneficial for all members, in that it increases their expected lifetime. Our aim is to explore this question.

2.1. **The state process in absence of cooperation.** We first consider a population where no lending and borrowing relations take place. The state process $\ell(k) = (\ell_t(k), t \geq 0)$ of a generic entity k in this population is a stochastic process taking values in the set $\{-1, 0, +1\}$, where

- state -1 indicates a shortage of ressource or distress,
- state 0 indicates a normal level of the resource,
- state $+1$ indicates an excess of resource.

For simplicity, we will refer to "survival" state when an entity is in one of the states $\{0, +1\}$, hence not distressed.

We consider the population is made of independent and homogeneous entities, meaning, the processes $\ell(i)$ and $\ell(j)$with $i \neq j \in \mathbb{N}^*$ are independent and have the same distribution. The state process of an entity is supposed to be a Markov chain with transition probabilities characterised by the following transition matrix:

$$\mathbf{Q}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & -(\alpha+\gamma) & \gamma \\ 0 & \delta & -\delta \end{pmatrix}, \tag{1}$$

where $\alpha > 0$, $\gamma > 0$ and $\delta > 0$ are constants, the elements of the matrix containing the rates at which the process transitions from one state state to another. The exponent for $\mathbf{Q}^{(1)}$ is to emphasise that the matrix characterises the time evolution of a single entity and in absence of cooperation (that is, a coalition of size 1). Later on, we will use $\mathbf{Q}^{(N)}$ for the transition matrix corresponding to a coalition of size N.

To completely specify the model for one entity without cooperation, we define a measurable space $(\Omega^{(1)}, \mathcal{F}^{(1)})$, where $\Omega^{(1)}$ is the space of right-continuous with left limits paths $\omega : \mathbb{R}_+ \to \{-1, 0, +1\}$ and $\mathcal{F}^{(1)}$ is the sigma algebra generated by these paths.

A special role will be played by the following distribution on $\boldsymbol{\mu} : \{-1, 0, +1\} \to [0, 1]$, defined as:

$$\boldsymbol{\mu}(-1) = p, \quad \boldsymbol{\mu}(0) = (1-p)c, \quad \boldsymbol{\mu}(+1) = (1-p)(1-c), \tag{2}$$

with

$$c := \frac{\delta}{\gamma+\delta}, \text{ and } p \in [0, 1].$$

Let $\mathbb{P}^{(1)} : (\Omega^{(1)}, \mathcal{F}^{(1)}) \to [0, 1]$ be the probability measure that ensures that an entity k has the initial distribution $\boldsymbol{\mu}$, that is $\mathbb{P}^{(1)}[\ell_0(k) = i] = \boldsymbol{\mu}(i),$ for all $i \in \{-1.0, +1\}$.

The probability $\mathbb{P}^{(1)}$ is known under the name of called *quasi-stationary distribution*. We refer to Méléard and Villemonais [17] for more details and an account of the relevance of this notion in the study of evolution of populations before extinction. From an economical perspective, an interpretation is that individual entities have reached an equilibrium between the production and depletion of the resource: before they become extinct, they are able to keep a constant probability

Key parameters:

- rate of the distress arrival (in absence of a ressource surplus)	$\alpha > 0$
- rate of surplus creation (in absence of a ressource surplus)	$\gamma > 0$
- rate of surplus depletion	$\delta > 0$
- probability of being distressed for a random entity (measure of adverse selection)	$p \in [0, 1]$
- probability of being in the neutral state given survival	$c = \frac{\delta}{\gamma+\delta} \in (0, 1)$
- viability ratio	$\xi = \frac{\gamma}{\alpha} > 0$

TABLE 1. Parameters characterising a random entity in absence of cooperation

of having a surplus, without dependence on the time frame. This property can be easily checked, and we state it below:

Lemma 2.1. *Consider the state process $\ell(k)$ of a given entity k in absence of cooperation. Under $\mathbb{P}^{(1)}$, it has a stationary conditional distribution given the entity is not distressed: for $t \geq 0$*

$$\mathbb{P}^{(1)}[\ell_t(k) = i | \ell_t(k) \neq -1] = \bar{\boldsymbol{\mu}}(i),\ \forall i \in \{-1, 0, +1\} \tag{3}$$

with $\bar{\boldsymbol{\mu}} : \{-1, 0, +1\} \to [0, 1]$ given by:

$$\bar{\boldsymbol{\mu}}(-1) = 0, \quad \bar{\boldsymbol{\mu}}(0) = c, \quad \bar{\boldsymbol{\mu}}(+1) = 1 - c. \tag{4}$$

Proof. See Appendix A.1. □

The process characterised in (1) has the feature that it will be absorbed in finite time in the state -1, hence it becomes distressed and extinct. While an entity is distressed, there are possibly other entities with a surplus. This raises the question of an allocation of the surplus of some entities toward other, distressed entities, with the scope of enabling them to recover from the distress state. In return, the entities ceding their surplus at one time may in the future benefit from the surplus of other entities in case of own distress. This is the basic idea of cooperation for survival that we want to develop. One question arises naturally: under what circumstances and to what extent is this mechanism of cooperation increasing the expected lifetimes of the cooperating entities ?

To answer this question, it is necessary to first introduce a model for the effect of cooperation in a group of fixed size $N < \infty$. This will be done in the next subsection. Afterwards, we will study the limiting behaviour when $N \to \infty$ and finally, the size of a coalition will be derived endogenously, using a game-theoretic approach.

2.2. **Cooperation within finite groups.** We generalise the previous model to describe the time evolution of the need of resource for members of coalitions, more exactly when $N \geq 1$ entities cooperate for survival. Hence, by taking $N = 1$ we recover the model without cooperation.

We shall call a coalition of size N an N-coalition. For defining its corresponding state process, we consider the probability space $(\Omega^{(N)}, \mathcal{F}^{(N)}, \mathbb{P}^{(N)})$, where $\Omega^{(N)} := (\Omega^{(1)})^N$, $\mathcal{F}^{(N)}$ is the corresponding product sigma-algebra and $\mathbb{P}^{(N)}$ is the product measure $(\mathbb{P}^{(1)})^N$.

We define a multivariate stochastic process $\mathbf{L}^{(N)}$, hereafter named the state process of the N-coalition, that keeps track of the resource shortage or availability in a group of size N. More exactly,

$$\mathbf{L}^{(N)} := (L_t^{(N)}(1), \cdots, L_t^{(N)}(N)), t \geq 0,$$

with each component $L_t^{(N)}(k)$ taking one of the values $\{-1, 0, +1\}$ at time $t \geq 0$ and representing the state of member k of the coalition. For any member, we have an interpretation for the states as surplus, neutral and distress (same notation prevails as in the absence of cooperation, see Subsection 2.1).

The type of cooperation we aim at modelling can be described as follows. Let us take an element $\mathbf{x} = (x(i), i \in \{1 \cdots, N\}) \in \{-1, 0, +1\}^N$ and consider that at time $t \geq 0$ we have $\mathbf{L}_t^{(N)} = \mathbf{x}$. If member i is being distressed (that is, $x(i) = -1$), then we assume it can borrow from any member in the group that has a surplus, that is every member j with $x(j) = +1$. If there are several such possible lenders, we shall assume that one is chosen randomly among them. Once it is able to borrow, the state of entity i turns from -1 to 0, and the state of its lender turns from +1 to 0. If there is no entity with a surplus at that time, then there is no potential lender and member i remains distressed. At a future time, members of the coalition may reach a surplus state and in this case lending and borrowing relations will resume, between one entity with a surplus and one distressed entity. We assume that except these borrowing lending relations there are no other interactions between the members of the coalition. Also, we will assume that entities are in an equilibrium as long as there are no distressed entities in the group (see Condition 2 below). The remaining of this subsection is dedicated to a rigorous mathematical formalisation of this mechanism of cooperation taking place between members of a coalition of size N.

First thing is to notice is that the mechanism we just described prevents an entity from being distressed as long as there are entities with surplus in the coalition, or, equivalently, an entity will persist in the distressed state -1 only as long as there are no entities with surplus in the coalition. The process $\mathbf{L}^{(N)}$ needs to satisfy:

If there exists $k \in \{1, ..., N\}$ such that $L_t^{(N)}(k) = -1$, then $\mathbf{L}_t^{(N)} \in \{-1, 0\}^N$.

That means that the liquidity process $\mathbf{L}^{(N)}$ is not visiting all the states in $\{-1, 0, +1\}^N$. This is formalised below in Condition 1.

Condition 1. ***The state space of the process.*** The state space of the process $\mathbf{L}^{(N)}$ is:

$$I := \{\mathbf{x} \in \{-1, 0, +1\}^N \mid \text{ if } \exists i \text{ with } x(i) = -1 \text{ then } x(j) \in \{-1, 0\} \; \forall j \in \{1, \cdots, N\}\}. \quad (5)$$

The next condition is the generalisation of the one for a single entity:

Condition 2. ***Distribution of the state process conditional on "no distress".*** As long as there are no distressed members, that is $\mathbf{L}^{(N)} \in \{0, +1\}^N$, the distribution of the process $\mathbf{L}^{(N)}$ is stationary: there exists a probability $\bar{\boldsymbol{\mu}}^{(N)}$ on $\{-1, 0, +1\}^N$ such that for all $A \in \{-1, 0, +1\}^N$:

$$\mathbb{P}^{(N)}(\mathbf{L}_t^{(N)} \in A \mid L_t(k) \in \{0, +1\} \; \forall k) = \bar{\boldsymbol{\mu}}^{(N)}(A).$$

The mathematical definition of the liquidity process $\mathbf{L}^{(N)}$ is given in Definition 2.5 below. But before we get there, let us already introduce the main objects involved:

- the interaction times, i.e., when cooperation occurs,
- the evolution of the state process between these interaction times,
- the concrete effect of cooperation when it occurs.

The *interaction times* are defined as a sequence of stopping times $(\theta(n), n \geq 1)$, representing the ordered times when interaction occurs within the group. As explained already, interaction materialises in one distressed entity being rescued by another entity, member of the coalition, having a surplus. We assume that these are the only interactions between the components of the process $\mathbf{L}^{(N)}$ and in between these times, i.e., for $t \in [\theta(n), \theta(n+1))$, the components of the vector process $\mathbf{L}^{(N)}$ are evolving independently one from another.

Consequently, the *law of the state process between any two successive transaction times*, is described by a stochastic process $\boldsymbol{\ell} = (\ell_t(1), \cdots, \ell_t(N))_{t \geq 0}$ taking values on $\{-1, 0, +1\}^N$ with components being independent Markov chains, with the evolution of any $\boldsymbol{\ell}(k)$, component k of $\boldsymbol{\ell}$, given by the transition matrix $\mathbf{Q}^{(1)}$ in (1). We shall use the notation $\boldsymbol{\ell}^{\mathbf{x}}$ for the process $\boldsymbol{\ell}$ satisfying the initial condition $\boldsymbol{\ell}_0 = \mathbf{x}$, for some $\mathbf{x} \in \{-1, 0, +1\}^N$.

The *effect of cooperation* is to reverse the state of distress of an entity, whenever surplus is available (i.e.,whenever there are entities in the surplus state within the coalition). Therefore, the process $\boldsymbol{\ell}$ will be reflected back into the set I whenever it exits. The successive exit times from I are precisely the transaction times $(\theta(n))$ mentioned above. It remains to define the reflection functions, which specify the state of the process $\mathbf{L}^{(N)}$ after transactions occur. This will be done in Definition 2.4 below.

Now, some additional notation:

Notation 2.2. *We introduce the following functions, defined on $\{-1, 0, +1\}^N$ with values in $\{0, \cdots, N\}$:*

$\eta^-(\mathbf{x}) := |\{i : x(i) = -1\}|$, *that is, the number of distressed entities, when the state of the coalition is* $\mathbf{x}$

$\eta^+(\mathbf{x}) := |\{i : x(i) = +1\}|$, *that is, the number of entities with a surplus, when the state of the coalition is* $\mathbf{x}$

where we denote by $|A|$ the cardinality of the set A, for A a countable set.

Notation 2.3. *We partition the set $E := \{-1, 0, +1\}^N \setminus I$ in two disjoint sets as follows:*

(i) The states where there are at least as many entities with a surplus as entities in distress, with at least one entity in distress:

$$E_+ := \{\mathbf{x} \in \{-1, 0, +1\}^N \mid \eta^+(\mathbf{x}) \geq \eta^-(\mathbf{x}) \geq 1\}.$$

(ii) The states where there are more distressed entities than entities with a surplus, with at least one entity with surplus:

$$E_- := \{\mathbf{x} \in \{-1, 0, +1\}^N \mid \eta^-(\mathbf{x}) > \eta^+(\mathbf{x}) \geq 1\}.$$

Definition 2.4 (The reflecting functions)**.** *We define a family of functions $(\mathcal{R}_n)_{n \geq 0}$ as follows. For any $n \geq 1$, the function $\mathcal{R}_n(\mathbf{x}) = (\mathcal{R}_n(\mathbf{x})(1), \cdots, \mathcal{R}_n(\mathbf{x})(N))$*

$$\mathcal{R}_n : E \times \Omega \to I$$

associates with an element $\mathbf{x} \in E$ an I valued random variable as follows:

(i) If $\mathbf{x} \in E_+$, then

$$\mathcal{R}_n(\mathbf{x})(k) = B_{n,k}, \quad k \in \{1, \cdots, N\},$$

where $(B_{n,k})_{k\in\{1,\cdots,N\},n\geq 0}$ is a family of independent and identically distributed random variable Bernoulli distributed with parameter $(1-c)$, where we recall that $c = \frac{\delta}{\gamma+\delta}$.

(ii) If $\mathbf{x} \in E_-$, then

$$\mathcal{R}_n(\mathbf{x})(k) = \begin{cases} 0 & \text{for } k \in \{i : x(i) = +1\} \cup C_n(\mathbf{x}), \\ \mathbf{x}(k) & \text{for } k \in \{i : x(i) = 0\} \cup \{i : x(i) = -1\} \setminus C_n(\mathbf{x}), \end{cases}$$

where for any $\mathbf{x} \in E_-$, $(C_n(\mathbf{x}))_{n\geq 0}$ is a family of independent random variables, where $C_n(\mathbf{x})$ is uniform on the set of all subsets of $\{i : x(i) = -1\}$ having precisely $\eta^+(\mathbf{x})$ elements.

The interpretation of the reflection functions is in line with our previous description of cooperation. In E_+, there are more entities with surplus than distressed entities. Therefore transfers of the resource may occur and allow all distressed entities to recover. The reflection function applied to elements in E_+ attributes only values 0 and $+1$ to all entities, so that no entity will be in distress. Importantly, the values are chosen so that Condition 2 is verified (as proved in Lemma 2.6 below).

When $\mathbf{x} \in E_-$, there are more distressed entities than entities with a surplus. An entity with a surplus will randomly choose a distressed entity and a transfer of the resource takes place; after the transfer, both entities reach the neutral state 0. The set of all distressed entities that will recover in given by $C_n(\mathbf{x})$, with cardinal $\eta^+(\mathbf{x})$, containinf randomly selected elements among the set of all distressed entities. We observe that not all distressed entities can be rescued in E_-. There will be precisely $\eta^-(\mathbf{x}) - \eta^+(\mathbf{x}) \geq 1$ distressed entities that cannot recover and they remain in the state -1; these are the elements of the set $\{i : x(i) = -1\} \setminus C_n(\mathbf{x})$. The neutral entities (that are in state 0) will keep their state 0 unchanged when applying the reflecting function.

Definition 2.5 (The liquidity process)**.** *The liquidity process of the N-coalition is a stochastic process denoted $\mathbf{L}^{(N)}$ taking values in the set:*

$$I := \{\mathbf{x} \in \{-1, 0, +1\}^N \mid \min\{\eta^-(\mathbf{x}), \eta^+(\mathbf{x})\} = 0\},$$

with:

$$\mathbf{x}_0 = \mathcal{R}_0(\boldsymbol{\ell}_0) \quad \text{for an initial state } \boldsymbol{\ell}_0 \in \{-1, 0, +1\}^N.$$

We denote $\theta(0) = 0$ and for $n \in \mathbb{N}$, we define $\mathbf{L}^{(N)}$ recursively:

$$\mathbf{L}_t^{(N)} = \boldsymbol{\ell}_{t-\theta(n)}^{\mathbf{x}_n} \text{ for } t \in [\theta(n), \theta(n+1)), \tag{6}$$

where:

$$\theta(n+1) := \theta(n) + \inf\{t \geq 0 \mid \boldsymbol{\ell}_t^{\mathbf{x}_n} \in E\}, n \geq 0, \text{ and} \tag{7}$$

$$\mathbf{x}_{n+1} := \mathcal{R}_{n+1}\left(\boldsymbol{\ell}_{\theta(n+1)}^{\mathbf{x}_n}\right). \tag{8}$$

Above, the quantity $\mathcal{R}_{n+1}\left(\boldsymbol{\ell}_{\theta(n+1)}^{\mathbf{x}_n}\right)$ defines the state in which the process $\mathbf{L}^{(N)}$ finds itself after the transaction $n+1$, taking place at time $\theta(n+1)$. The random functions $(\mathcal{R}_n)_{n\geq 0}$ each associates to a state in E a state in I (see Definition 2.4).

Note that there is a sequence of reflecting functions, and not only one reflecting function, and we used a unique reflecting function $\mathcal{R}_n$ per transaction $\theta(n)$. This way, there there will be independence of the liquidity process of the previously visited states, by using an independent random variable at each time to allocate the resource among the coalition members. We emphasise that nevertheless, for our analysis later on (next sections), we only need the distribution of the entities after a transaction take place, that is, so that we could also use one single reflecting function in our definition, with all the analysis that follows being still valid.

Also, we can notice that the state space of the process $\mathbf{L}^{(N)}$ satisfies Condition 1. We now verify that Condition 2 is as well satisfied.

Lemma 2.6. *The process $\mathbf{L}^{(N)}$ from Definition 2.5 satisfies Condition 2, namely, for all $t \geq 0$ and for all $A \in \{-1, 0, +1\}^N$:*

$$\mathbb{P}^{(N)}\left(\mathbf{L}_t^{(N)} \in A \mid \eta^-(\mathbf{L}_t^{(N)}) = 0\right) = \bar{\boldsymbol{\mu}}^{(N)}(A) \tag{9}$$

and with $\bar{\boldsymbol{\mu}}^{(N)} = \bar{\boldsymbol{\mu}}^N$ with $\bar{\boldsymbol{\mu}}$ defined in (4).

Proof. See Appendix A.2. □

3. EXPECTED LIFETIME WITH COOPERATION AND OPTIMAL SIZE OF COALITIONS

3.1. **Probabilistic properties of the distress process in an N-coalition.** We are now interested in computing the expected lifetime of an entity as a member of an N-coalition. This can be a complicate object to compute, because the state process $\mathbf{L}^N$is a multivariate process with interdependent components. But it turns out that the analysis becomes much easier, once we formulate it in the right setting. Our approach is to replace the state process $\mathbf{L}^N$ by a much simpler object, that only keeps track of the number of distressed entities at any point in time, and without the identities of these distressed entities. As we shall see below, this process nicely captures all the features that we need in our analysis, in particular the phenomena of cooperation.

Below, we introduce the stochastic process that keeps track of the distressed entities at any point in time:

Definition 3.1. *(The distress process). The distress process of an N-coalition denoted as $Y^{(N)}$, is is counting the number of distressed entities at time t within the N-coalition, that is:*

$$Y_t^{(N)} := |\{k \in \{1, \cdots, N\} \mid L_t^{(N)}(k) = -1\}|.$$

The distress process has the very convenient feature of being a Markov process and its distribution is characterised as follows:

Theorem 3.2. *The distress process $Y^{(N)}$ of an N-coalition is a Markov chain with state space $\{0, 1, ..., N\}$ and transition matrix $\mathbf{Q}^{(N)} = (q^{(N)}(i,j))$, defined as:*

$$q^{(N)}(i,j) = \begin{cases} \alpha N c^N & \text{if } i = 0 \text{ and } j = 1, \\ \alpha(N-i) & \text{if } j = i+1 \text{ and } i = 1, \cdots N, \\ \gamma(N-i) & \text{if } j = i-1 \text{ and } i = 1, \cdots N, \\ 0 & \text{otherwise, with } i \neq j. \end{cases}$$

The diagonal elements are as usual $q^{(N)}(i,i) = -\sum_{j\neq i} q^{(N)}(i,j)$.

Proof. See Appendix A.3. □

Further, we are going to take the distress process as a state variable in our analysis of coalitions. First thing, we can now characterise the expected lifetime:

Definition 3.3. *The expected lifetime of a member* i *in an N-coalition containing* $n \leq N$ *distressed members, is denoted it by* $h(n,N)$ *and is defined as:*

$$h(n,N) := \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{L_t^{(N)}(i)\in\{0,1\}\}} dt | Y_0^{(N)} = n\right] \tag{10}$$

for any $i \in \{1,\cdots,N\}$.

Remark 3.4. *Denote by* $\mathcal{F}_t^{Y^{(N)}} := \sigma(Y_s^{(N)}, s \leq t)$, *that is, the information about the process* $Y^{(N)}$ *up to time* t. *Under* $\mathbb{P}^{(N)}$ *all entities in the coalition have the same distribution conditionally on the filtration* $(\mathcal{F}_t^Y)$, *that is, the conditional distribution of* $L_0^{(N)}(i)$ *and* $L_0^{(N)}(j)$ *are identical. For this reason, the expected lifetimes of all entities in a coalition are equal.*

Theorem 3.5. *The expected lifetime of entity* i *given that* $Y_0 = n$ *is as follows:*

(a) *if* $\alpha \neq \gamma$, *then*

$$h(n,N) = h^*(n/N) + \frac{1}{\alpha N c^N}\left(1 - \frac{c^N}{1-\xi}\right)\frac{\xi^n - \xi^N}{1-\xi}, \tag{11}$$

where $h^*(z) := \frac{1-z}{\alpha-\gamma}$ *and* $\xi := \frac{\gamma}{\alpha}$;

(b) *if* $\alpha = \gamma$, *then*

$$h(n,N) = \frac{1-n/N}{\alpha}\left(\frac{1}{c^N} + \frac{N+n-1}{2}\right), \tag{12}$$

Proof. See Appendix A.4. □

3.2. **Optimal size of coalitions.** Let us suppose that a central planner can decide on the size of the cooperating groups and will choose a group size N^* (possibly infinite) that maximises the expected lifetimes of member entities. We also suppose that the central planner does not observe the state of an entity and choses randomly the entities from the entire population to form a coalition. When the size of a coalition is N, we assume that the distress process will satisfy $Y_0^{(N)} \sim \mathrm{Bin}(N,p)$, that is a binomial random variable with parameters N and p, that counts the number of distressed entities in an N-coalition at time $t = 0$. We assume that $p < 1$, since otherwise any N-coalition is extinct at time $t = 0$ and has a null expected lifetime. Under this assumption, we compute the *optimal expected lifetime* as:

$$H^* := \sup_{N\geq 1} \mathbb{E}\left[h(Y_0^{(N)}, N)\right]$$

and we obtain:

Proposition 3.6. *The following hold:*

(a) *If either* $\frac{\gamma}{\alpha} \geq c$ *with* $p \in [0, 1)$*, or* $\frac{\gamma}{\alpha} \in [0, c)$*, with* $p < \frac{1-c}{1-\frac{\gamma}{\alpha}}$ *then the optimal expected lifetimes satisfies:*

$$H^* = \lim_{N\to\infty} \mathbb{E}\left[h(Y_0^{(N)}, N)\right] = \infty,$$

that is, the optimal coalition size is $N^* = \infty$*.*

(b) *If* $\frac{\gamma}{\alpha} \in [0, c]$*, with* $p \geq \frac{1-c}{1-\frac{\gamma}{\alpha}}$*, then both the optimal expected lifetime* H^* *and the optimal coalition size is* N^* *are finite. Further, it holds that*

$$H^* \geq \lim_{N\to\infty} \mathbb{E}\left[h(Y_0^{(N)}, N)\right] = h^*(p) = \frac{1-p}{\alpha - \gamma} < \infty.$$

Proof. See Appendix A.5. □

In Proposition 3.6, we characterise the parameters of populations where it is optimal cooperate in infinite groups ($N^* = \infty$), versus finite groups ($N^* < \infty$). When $\frac{\alpha}{\gamma} \geq c$ infinite groups are always optimal, independently of the value of the parameter p. This means that as long that there is a non null fraction of surviving individuals to start with (given by $1 - p$), even very small, it is possible for the distressed entities to recover trough cooperation and achieve an infinite expected lifetime. But when $\frac{\alpha}{\gamma} < 1$, the situation is different. The expected lifetime may be finite, if the fraction of distressed members is high initially at time $t = 0$, and exceeds a certain threshold. Illustrations are provided in Figures 1 and 2, where we can observe (for the case $\frac{\gamma}{\alpha} \leq c$), both situation of convergence and divergence to infinity, of the expected lifetime of N-coalitions with N increasing.

A much finer understanding of the previous result is obtained when computing explicitly the time evolution of the proportion of distressed entities in infinite coalitions. This is analysed below.

Theorem 3.7. *The fraction of distressed entities in an N-coalition at time* t*:*

$$Z_t^{(N)} := \frac{Y_t^{(N)}}{N}$$

has the property:

$$\lim_{N\to\infty} Z_t^{(N)} = Z_t,$$

where

$$Z_t = 1 - (1 - Z_0)e^{(\gamma-\alpha)t\wedge\tau}$$

with $\tau = \inf\{t \mid Z_t = 0\}$*. Consequently, if* $Z_0 = 0$ *then* $Z \equiv 0$*. Otherwise if* $Z_0 > 0$*, we have:*

(i) *If* $\frac{\gamma}{\alpha} > 1$*, then the process is absorbed at 0 after a deterministic time* $\tau < \infty$*.*

(ii) *If* $\frac{\gamma}{\alpha} = 1$*, then* $Z \equiv Z_0$*.*

(iii) *If* $\frac{\gamma}{\alpha} < 1$*, then* $\lim_{t\to\infty} Z_t = 1$*.*

Considering still that $Y_0^{(N)} \sim \text{Bin}(N, p)$, we obtain by the strong law of large numbers $Z_0 = p$. Consequently, if $\frac{\gamma}{\alpha} > 1$, by the finite time τ all distressed entities have recovered and there will be no more distressed individuals in the population ever after. The expected lifetime is indeed infinite in this case. The case $\frac{\gamma}{\alpha} = 1$ corresponds to the situation where an infinite population will keep through time a constant proportion of distressed entities. Any distressed entity may recover and become distressed again later on, with a constant probability through time of being in one of these

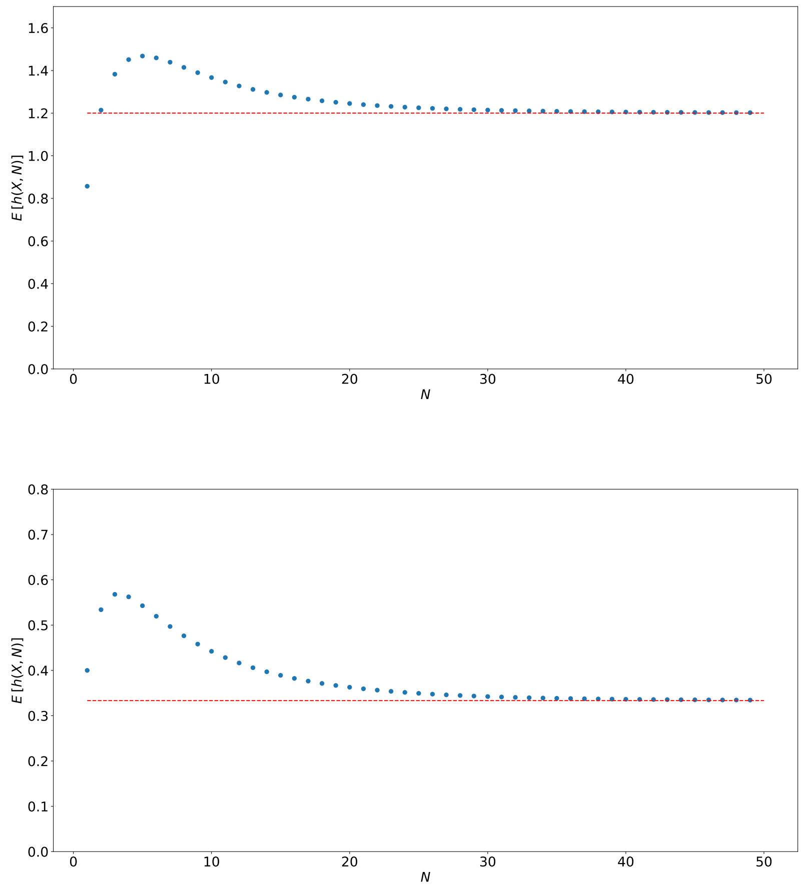

FIGURE 1. Convergence of the expected lifetime when the size of the coalition N increases for $\frac{\gamma}{\alpha} \in [0, c]$ and $p \geq \frac{1-c}{1-\frac{\gamma}{\alpha}}$. The horizontal red line is the level $h^*(p) = \frac{1-p}{\alpha-\gamma}$. Above: $\alpha = 0.5$, $\frac{\gamma}{\alpha} = 0.4$, $c = 0.5$, $p = 0.4$. Below: $\alpha = 0.5$, $\frac{\gamma}{\alpha} = 0.4$, $c = 0.7$ and $p = 0.7$. The maximum expected lifetime H^* is obtained with corresponding group sizes $N^* = 5$ (above) and $N^* = 3$ (below).

two situations. Finally, the case $\frac{\gamma}{\alpha} < 1$ corresponds to the situation where the infinite coalitions become extinct in the limit, as $t \to \infty$. This complements the understanding of Proposition 3.6, as we see that the parameter p does not change the final fate of such populations of becoming extinct asymptotically, nor it influences the rate of at which such populations decline (which is given by $\alpha - \gamma > 0$). Nevertheless, as p low enough, the entities can enjoy infinite expected lifetimes, as stated in Proposition 3.6 (a).

We may conclude that from the perspective of a central planner aiming at maximising expected lifetimes in populations, organising cooperation in the whole population is an optimal policy whenever the conditions of Proposition 3.6 (a) are fulfilled, while cooperation in smaller sub-groups,

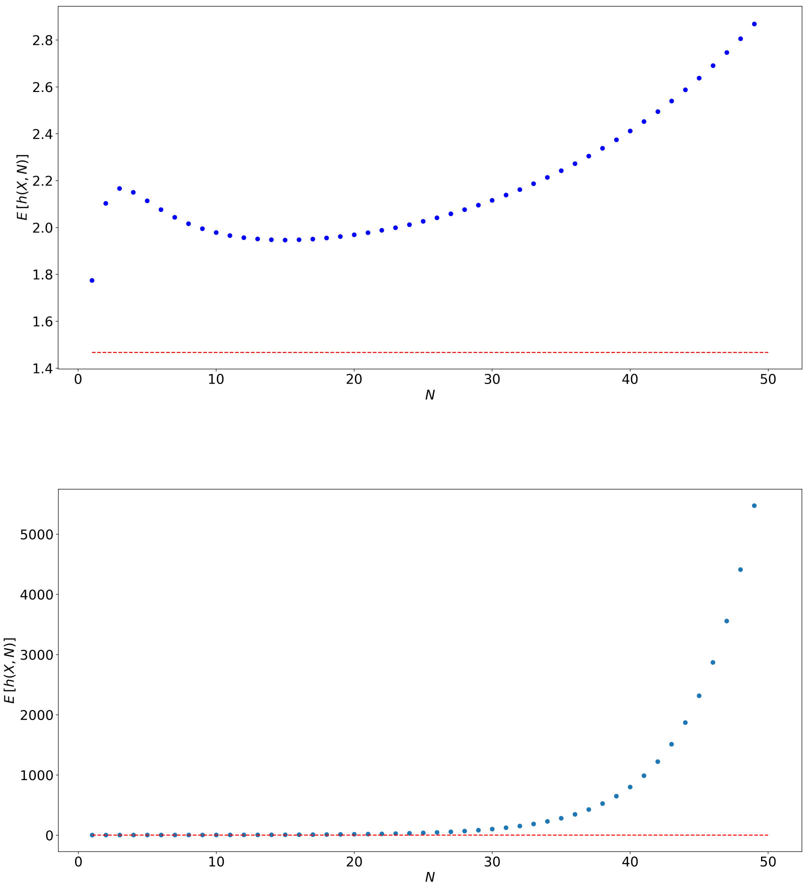

FIGURE 2. Divergence of the expected lifetime when the size of the coalition N increases for $\frac{\gamma}{\alpha} \in [0, c]$, with $p < \frac{1-c}{1-\frac{\gamma}{\alpha}}$. The horizontal red line is the level $h^*(p) = \frac{1-p}{\alpha-\gamma}$. Above: $\alpha = 0.5$, $\frac{\gamma}{\alpha} = 0.25$, $c = 0.7$, $p = 0.45$. Below: $\alpha = 0.5$, $\frac{\gamma}{\alpha} = 0.2$, $c = 0.6$ and $p = 0.3$. The optimal size of coalitions N^* is infinite.

and prohibiting expanding the groups, is optimal whenever the conditions of Proposition 3.6 (b) are fulfilled.

A crucial aspect at his point is to understand the self-organisation process of entities in cooperating groups and whether or not this process reaches by itself an optimal level of cooperation, without exterior intervention, such of a central planner. This topic is investigated in the remaining of the paper.

4. Nash equilibria and stability of coalitions

We now turn to the study of the formation of coalitions within the frame of strategic game theory. This will allow us to understand to what extent the expected lifetimes will be improved when entities choose strategically to enter or not coalitions and coalitions also chose strategically whether to accept outsider entities and enlarge. We shall assume players choose strategies continuously in time, in order to achieve the highest possible expected lifetime for themselves.

We shall define the stable coalitions as those Nash equilibria, resulting in a coalition size remaining stable through time, independently of the state of distress of its members. Stable coalitions are those that have reached a certain size and their corresponding decision process will exclude new members ever after. Our objective is to identify those structures which the players do not wish to upset-since they have no better alternatives. However, in what sense should the expression "better alternative" be understood? This will be explored in this section.

4.1. **Formalisation of the game.** The game is represented by a sequence of stochastic non cooperative games $\Gamma = (\Gamma(N))_{N\geq N_0}$, with N_0 being a natural number. The stochastic games are played sequentially, one after the other: when $\Gamma(N)$ ends, $\Gamma(N+1)$ starts, unless the the game $\Gamma(N)$ never ends, in which case the subsequent games will not be played. The game $\Gamma(N_0)$ starts at time $t = 0$. The variable N of $\Gamma(N)$ characterises the size that the coalition has reached, so that N_0 is the initial size of the coalition.

The game $\Gamma(N)$ corresponds to a game between a coalition of size N and the "outside world", synthesised in player $\mathcal{P}_\ominus$, representing the proportion p of distressed entities, and player $\mathcal{P}_\oplus$, representing the proportion $(1-p)$ of not distressed entities. Therefore, at any point in time there are three players to be modelled only. Even though the outside world contains an infinity of individuals, they only are of two types and it is sufficient to model only one representative of each type. For simplicity, we always consider $p \in (0,1)$, so that we do have the three players indeed.

All players choose actions continuously in time, while observing the state process of the N-coalition, $Y_t^{(N)}$, at any time t for the duration of the game. So, the state process for $\Gamma(N)$ is the Markov process $Y^{(N)}$. The N-coalition will expand to reach the size $N+1$ when the N-coalition decides to include a new member; this new member will be chosen randomly among all those that played the action to accept to enter the N-coalition at that point in time. Because the new member is randomly chosen, it is implicitly assumed that the N-coalition faces uncertainty regarding the type of the new member whenever both players $\mathcal{P}_\oplus$ and $\mathcal{P}_\ominus$ are playing the strategy to join the coalition. This is the typical adverse selection problem (see Akerloff [1]). Indeed, with perfect information, the coalition would privilege the type $\mathbb{P}_\oplus$, so that the new member would not be randomly chosen. As soon as a new member is added, the game $\Gamma(N)$ stops and the game $\Gamma(N+1)$ continues from that point in time, with players being the coalition of size $N+1$ and the outside world, in a similar manner.

To make things precise, we follow the formalisation of a continuous time stochastic game introduced in Neyman [20]. The stochastic game $\Gamma(N)$ corresponding to the size N of the coalition is defined as follows:

$$\Gamma(N) = \left(\{\text{N-coalition}, \mathcal{P}_\oplus, \mathcal{P}_\ominus\}, \mathcal{I}^{(N)}, \mathcal{A}, \mathcal{H}^{(N)}, P^{(N)}, n_0^{(N)}\right),$$

where:

- $\{\text{N-coalition}, \mathcal{P}_\oplus, \mathcal{P}_\ominus\}$ is the set of players.
- $\mathcal{I}^{(N)} = \{0, 1, \cdots, N\} \times \{0\} \times \{1\}$ is the state space of the players.
- $\mathcal{A} = \{a, b\}^3$ is the set of player's strategies (or actions). The set of actions remains the same, regardless of the size N or state n of the coalition.
- $\mathcal{H}^{(N)} : \{0, 1, \cdots, N\} \times \mathcal{A} \times \{\text{N-coalition}, \mathcal{P}_\oplus, \mathcal{P}_\ominus\} \to \mathbb{R}_+$ is the payoff function, with values provided in the Table 1 below. Notice the use of a random variable $e(N)$ that that stands for the type of the new member of the group, should the N-coalition extend by action (a, a, a). More details on this will follow below.
- $P^{(N)} = (\mathbf{Q}^{(N)}, \delta_{\{0\}}, \delta_{\{1\}})$ are the transition probabilities of the three players. The transition matrix $\mathbf{Q}^{(N)}$ (that is given in Theorem 3.2) corresponds to the state process of the N-coalition, the state of $\mathcal{P}_\oplus$ is constantly zero, and the state of $\mathcal{P}_\ominus$ is constantly one (by convention, so that the state 1 indicates distress).
- $n_0^{(N)} \in \{0, \cdots, N\}$ is the initial distribution for the N-coalition, at the time when the game $\Gamma(N)$ starts. This means that the state process $Y^{(N)}$ of the N-coalition takes an initial value $n_0^{(N)}$ at the time when the game $\Gamma(N)$ is initiated. The initial value will be endogenously determined within the game, except for the very first game $\Gamma(N_0)$, where it is exogenously specified. This will be further detailed below.

Payoff function $\mathcal{H}^{(N)}(n, s, i)$			
Actions	*Player i*		
$s \in \mathcal{A}$	$i =$ N-coalition	$i = \mathcal{P}_\oplus$	$i = \mathcal{P}_\ominus$
(a, a, a)	$\mathbb{E}\left[h(n + e, N + 1)\right]$	$\mathbb{E}\left[h(n, N + 1)\mathbb{1}_{\{e=0\}} + h(0, 1)\mathbb{1}_{\{e=1\}}\right]$	$\mathbb{E}\left[h(n + 1, N + 1)\mathbb{1}_{\{e=1\}}\right]$
(a, a, b)	$h(n, N + 1)$	$h(n, N + 1)$	0
(a, b, a)	$h(n + 1, N + 1)$	$h(0, 1)$	$h(n + 1, N + 1)$
(a, b, b)	$h(n, N)$	$h(0, 1)$	0
(b, a, a)	$h(n, N)$	$h(0, 1)$	0
(b, a, b)	$h(n, N)$	$h(0, 1)$	0
(b, b, a)	$h(n, N)$	$h(0, 1)$	0
(b, b, b)	$h(n, N)$	$h(0, 1)$	0

TABLE 2. Values of the payoff function $\mathcal{H}^{(N)}(n, s, i)$, where $n \in \{0, \cdots, N\}$ is fixed. We consider $e \sim \text{Bernoulli}(p)$, with $p \in (0, 1)$.

We introduce the following additional rules for the game $\Gamma(N)$, where $N \geq N_0$:

(i) *Strategy process.* We suppose each player chooses actions among $\{a, b\}$, continuously in time, where a stands for "accept" (agree to cooperate and extend the coalition) and b for "block" (or not agree to extend the coalition). The decision (or strategy) process is a stochastic process $D_{t\in[T(N),T(N+1)]}$, adapted to the filtration generated by the state process $Y^{(N)}$, and taking values in $\mathcal{A} = \{a, b\}^3$, with D_t being the action at time t, $T(N)$ the starting time (with $T_N = 0$ if $N = N_0$) of the game and $T(N + 1)$ the stopping time of the game (defined below).

(ii) *Stopping rule.* When the N-coalition is not extinct (that is $Y_t^{(N)} < N$), the game $\Gamma(N)$ is ended whenever N-coalition plays a and at least one type of players in the outside world plays a. Whenever the N-coalition is extinct (that is $Y_t^{(N)} = N$), we consider the game $\Gamma(N)$ continues, unless player $\mathbb{P}_\oplus$ steps in to rescue (that is, N-coalition plays a and $\mathbb{P}_\oplus$ plays a). By this, we avoid studying enlargements of extinct coalitions that only lead to extinct coalitions. Therefore, the ending time of the game $\Gamma(N)$ is defined as:

$$T(N+1) := \inf\left\{t \geq T(N) \mid \text{ either: } Y_t^{(N)} < N, D_t \in E \text{ or: } Y_t^{(N)} = N, D_t \in E', \right\} \tag{13}$$

with $E = \{(a,a,a),(a,b,a),(a,a,b)\}$ and $E' = \{(a,a,a),(a,a,b)\}$. We use the convention $\inf \emptyset = +\infty$. Remark that $T(N)$ is a stopping time in the filtration generated by the state process $Y^{(N)}$.

(iii) *The payoff function.* Table 1 summarises the payoff function at time $t \in [T(N), T(N+1))$ for each player, depending on:
- the state of coalition at time t, given by the state of process $Y_t^{(N)} = n \in \{0,1,...,N\}$,
- the action taken by the players at time t, $D_t \in \mathcal{A}$.

As there are finitely many states, players and strategies, that table is exhaustive of the payoffs.

The payoffs make appear the function h that is the expected lifetime in coalitions. Whenever the actions are not in E, players have a payoff that equals their own expected lifetime. When actions belong in the set E, the coalition expands to include a new member and therefore the payoffs depend on the state of the new member $e(N) \in \{0,1\}$. This new member is assumed to be randomly chosen among the outside entities that play a. So, for actions belong in the set E, the payoff for the coalition and for the new accepted member is $h(n+e(N), N+1)$. For instance, if action (a,a,a) is played, the coalition chooses one new member at random from the general population, hence this new member will be distressed with probability p and not distressed with probability $1-p$. We obtain that $e(N)$ is Bernoulli with parameter p. As $e(N)$ is independent from the filtration of the process $Y^{(N)}$, but the decision D must be adapted to it, we project the payoff on the the filtration of the process $Y^{(N)}$. In this case, strategies are decided by players upon comparing expected values. In the cases where action (a,b,a) or (a,a,b) are played, only one type within the outside population is willing to join the N-coalition, so that $e(N)$ equals the state of this player (hence is not random).

(iv) *Starting rule.* At $t = T(N+1)$ the game $\Gamma(N+1)$ starts with initial value

$$n_0^{(N+1)} = Y_{T(N+1)}^{(N)} + e(N).$$

As the state process $Y^{(N)}$ is Markov, we will further focus on Markov strategies only, that are only dependent on the current state. That means that we will impose the stronger requirement that the strategy process $(D_t)_{t\in[T(N),T(N+1)}$ satisfies the property: $\forall t \geq 0$, D_t is a function of $Y_t^{(N)}$. Requiring the strategies to be functions of the current state of the process $Y^{(N)}$ is very useful, as it permits to analyse the game $\Gamma(N)$ through the collection of sub-games $\Gamma(n,N), n \in \{0,\cdots,N\}$, with each sub-game. $\Gamma(n,N)$ being a non-stochastic game, corresponding to payoff $\mathcal{H}^{(N)}(n,s,i)$, $s \in \mathcal{A}$ and $i \in \{\text{N-coalition}, \mathbb{P}_\oplus, \mathbb{P}_\ominus\}$.

A fundamental concept for non zero sum games is the Nash equilibrium. The definition below sates that a Nash equilibrium for $\Gamma(N)$ at time t, given that $Y_t^{(N)} = n$ is a Nash equilibrium for the sub-game $\Gamma(n, N)$.

Definition 4.1. *A Nash equilibrium in the stochastic game $\Gamma(N)$ at time t is an element $s \in \mathcal{A}$, that satisfies, for $n = Y_t^{(N)}$:*

$$\mathcal{H}^{(N)}(n, s, i) \geq \mathcal{H}^{(N)}(n, s^{-i}, i),$$

for any $i \in \{$N-coalition$, \mathbb{P}_{\oplus}, \mathbb{P}_{\ominus}\}$. The notation s^{-i}defines the action obtained by flipping the i^{th} coordinate of the action s to the alternative strategy (i,e. if it is a then it's flipped to b and vice versa) while maintaining the strategies of the other players unchanged.

We will dedicate the remaining of the paper to the study of the stable coalitions. The notion of stability is introduced below:

Definition 4.2. (i) *An N-coalition is stable if, when Nash equilibria strategies are played in each point in time, the game $\Gamma(N)$ is never ending (that is, $T(N+1) = \infty$).*
(ii) *A coalition that is not stable is said to be unstable.*

Remark 4.3. *Given the expression in (13), an N-coalition is stable if, whenever $Y_t^{(N)} < N$, there are not Nash equilibria in $E = \{(a, a, a), (a, b, a), (a, a, b)\}$ and, when $Y_t^{(N)} = N$, there are not Nash equilibria in $E' = \{(a, a, a), (a, a, b\}\}$*

4.2. **An analysis of stability of coalitions.** In order to analyse stability of coalitions, we will be assuming in this subsection that players choose their strategies only among the Nash equilibria, at each point in time and for any size N and state n of a coalition that are prevailing at that point in time. We will prove the existence of stable coalitions for some range of parameters α, γ, δ and p and will characterise some generic conditions for stability versus instability.

Below, we characterise stability of N-coalitions in terms of inequalities involving the payoff functions of the players.

Proposition 4.4. *An N-coalition is* stable *if the following conditions are fulfilled:*

(i) *for all $n \in \{0, \ldots, N-1\}$,*
$$h(n, N) > h(n+1, N+1); \tag{14}$$
(ii) *for all $n \in \{0, \ldots, N\}$ one of the following holds:*
$$h(0, 1) > h(n, N+1) \tag{15}$$
$$\textit{or} \quad h(n, N) > ph(n+1, N+1) + (1-p)h(n, N+1). \tag{16}$$

Proof. Using the values of the payoff function $\mathcal{H}^{(N)}$ written in Table 1, we need to prove that:

1. the condition (14) together with either (15) or (16) , for $n \in \{0, \ldots, N-1\}$ are equivelant to: there are not Nash equilibria in $E = \{(a, a, a), (a, b, a), (a, a, b)\}$ for the corresponding sub-game $\Gamma(n, N)$;
2. the condition either (15) or (16) , for $n = N$ is equivalent to: there are not Nash equilibria in $E' = \{(a, a, a), (a, a, b)\}$ for the corresponding sub-game $\Gamma(N, N)$.

First, we observe that (a, a, b) is never a Nash equilibrium for $n < N$. Intuitively, distressed players have always the incentive to enter a coalition and get a chance to be be rescued. Indeed, let us assume that the N-coalition plays a. One can notice by inspection of the payoff function $\mathcal{H}^{(N)}$, that as long as an N-coalition is not extinct, the player $\mathbb{P}_{\ominus}$ will always play a (as the corresponding payoffs strictly dominates the payoff 0, which would be obtained by playing b).

Condition (i) is the equivalent of: The action (b, b, a) delivers a strictly higher payoff for the N-coalition than (a, b, a) and this, for all $n \in \{0, 1..., N-1\}$, so that (a, b, a) is not a Nash equilibrium in the corresponding sub-games.[1]

Condition (ii) for $n \in \{0, \cdots, N-1\}$ is the equivalent of: either the action (b, a, a) delivers a strictly higher payoff for the N-coalition than action (a, a, a), or the action (a, b, a) delivers a higher payoff for $\mathbb{P}_{\oplus}$ than action (a, a, a). In either case, (a, a, a) is a not a Nash equilibrium for the sub-games $\Gamma(n, N)$, $n \in \{0, \cdots, N-1\}$.

Now, focussing specifically on condition (ii) and the case $n = N$ we find that (16) is not fulfilled. So that condition (ii) requires that (15) holds true. But this implies that player $\mathbb{P}_{\oplus}$ has a strict advantage in playing b. Consequently neither strategies in E' are Nash equilibria. $\square$

It is useful to take the point of view of the healthy outsiders, that is, player $\mathbb{P}_{\oplus}$. It will play a if $h(n, N+1) \geq h(0, 1) = \frac{1}{\alpha c}$ and b otherwise. Given that the mapping $n \mapsto h(n, N)$ is decreasing, there is a critical level of distress of an N-coalition beyond which healthy outsiders will not be willing to join and N-coalition. We call this level, the critical level of distress.

Definition 4.5 (Critical level of distress for an N-coalition)**.** *For $0 < N < \infty$ fixed, we define $n^{out}(N)$, as*

$$n^{out}(N) := \max \left\{ n \in \{0, \dots, N\} \mid h(n, N+1) \geq \frac{1}{\alpha c} \right\},$$

that is,

$$n^{out}(N) = \min \left\{ n \in \{0, \dots, N+1\} \mid h(n, N+1) < \frac{1}{\alpha c} \right\} - 1.$$

Notice that n^{out} is well-defined. Indeed, the mapping: $n \mapsto h(n, N+1)$ is decreasing (see Lemma A.2 in Appendix A.7), and therefore, we have: $h(0, N+1) \geq h(0, 1) = \frac{1}{\alpha c}$, so $n^{out}(N) \geq 0$. Moreover, $h(N+1, N+1) = 0 < \frac{1}{\alpha c}$, that proves the existence on n^{out} satisfying $n^{out}(N) \leq N$.

Lemma 4.6. *If $n^{out}(N) = N$, then an N coalition is unstable. Non distressed outsiders always want to enter such coalitions, regardless its of its state of distress.*

Proof. Suppose that the N coalition extinct that is, $n = N$, then $h(N, N) = 0$. Also, because $n^{out}(N) = N$ we have $h(N+1, N) > h(0, 1)$. Therefore, condition (ii) in Proposition 4.4 is not satisfied and the Nash equilibrium is $\{a, a, a\}$. The coalition extends to incorporate a new member. $\square$

[1]We do not impose the condition (i) for $n = N$ because we know that there it is not satisfied. We have indeed: $h(N, N) = h(N+1, N+1) = 0$. This means that an extinct N-coalition will be indifferent between action (a, b, a), that is, enlarging with a distressed entity, or action (b, b, a), that is, block the entrance of a distressed entity. But both actions lead to an extinct coalition.

In the next proposition we give a characterisation of a stable N-coalition in terms of $n^{out}(N)$, that will reduce the number of inequalities to check (indeed Proposition 4.4 contained some overlapping conditions).

Proposition 4.7. *An N-coalition is stable if and only if the following hold:*

(i) *for all* $n \in \{0, \ldots, n^{out}(N)\}$ *we have:*
$$h(n,N) > ph(n+1,N+1) + (1-p)h(n,N+1). \tag{17}$$

(ii) *If* $n^{out}(N) \leq N-2$*, then for all* $n \in \{n^{out}(N)+1, \ldots, N-1\}$ *we have:*
$$h(n,N) > h(n+1,N+1). \tag{18}$$

Proof. The assertions (i) and (ii) are direct consequences of Proposition 4.4 and Definition 4.5 . □

We now give some results about the stability of coalitions. The next result is general, as it does not involve the probability p that is, the proportion of distressed entities in the "outside world". It will be followed by an analysis involving the role or the probability p, in the next section.

Proposition 4.8. *The following assertions hold:*

(i) *If* $\frac{\gamma}{\alpha} \geq 1$*, then any N-coalition is unstable:* $n^{out}(N) = N$ *for all* N.
(ii) *If* $\frac{\gamma}{\alpha} \geq c$*, then there are not arbitrarily large stable N-coalitions: there exists* N^* *such that* $n^{out}(N) = N$ *for all* $N \geq N^*$.
(iii) *If* $\frac{\gamma}{\alpha} \in (0,c)$*, then* $n^{out}(N) < N$ *for all* N.

Proof. We use the representation (33) of the function h. We have:
$$h(N,N+1) = \frac{1}{\alpha c(N+1)} \left(\left(\frac{\frac{\gamma}{\alpha}}{c}\right)^N + \sum_{j=0}^{N-1} c\left(\frac{\gamma}{\alpha}\right)^j \right). \tag{19}$$

(i) By Lemma 4.6, to show that the N-coalition is not stable, it suffices to show that $h(N,N+1) \geq \frac{1}{\alpha c}$, that is $n^{out}(N) = N$ (the healthy outside entities are always willing to join the N-coalition). Using (19) and $\frac{\gamma}{\alpha} \geq 1$, we obtain:
$$h(N,N+1) \geq \frac{1}{\alpha c}\left(\frac{1+Nc^{N+1}}{(N+1)c^N}\right) > \frac{1}{\alpha c},$$
because $\frac{1+Nc^{N+1}}{(N+1)c^N} > 1$, this being equivalent to $c < 1$ (the condition on c in the beginning of Subsection 4.2).

(ii) If $\frac{\gamma}{\alpha} \geq c$, then $\lim_{N\to\infty} h(N,N+1) = +\infty$, so that there is N^* so that $h(N,N+1) > \frac{1}{\alpha c}$ for all $N \geq N^*$.

(iii) We have $n^{out}(N) < N$ if and only if $h(N,N+1) < \frac{1}{\alpha c}$. Or, this inequality can be checked easily, using $\frac{\gamma}{\alpha} < c$ in (19) :
$$h(N,N+1) < \frac{1}{\alpha c} \cdot \frac{N+1}{N+1} = \frac{1}{\alpha c}.$$

□

5. Some numerical illustrations

5.1. **Characterisation of the non-cooperative populations.** Whenever the 1-coalitions are stable, there are no incentives for individuals to form pairs for the purpose of cooperation, as cooperation with another entity reduces the expected lifetimes of not extinct entities. Indeed, it can be proved that if the probability of being distressed is above a certain threshold, entities that are in survival will not take any risk to cooperate, as the probability of having to pair with a distressed entity is too high as compared with the benefits of future cooperation (the adverse selection risk is simply too high in this case). We give below the exact threshold for the probability:

Proposition 5.1 (The stable 1-coalitions)**.** *For any c and $\frac{\gamma}{\alpha}$ fixed, there exists a critical probability:*

$$\pi_1 = \max\left\{\frac{\gamma}{\alpha} + (1-c)^2, 1\right\}$$

such that: if $\pi_1 < 1$, then for all $p > \pi_1$ the corresponding 1-coalition is stable and for all $p \leq \pi_1$ the corresponding 1-coalition is unstable. If $\pi_1 = 1$ then, for any $p \in (0,1)$, then stable coalitions do not exist.

Proof. We impose the conditions Proposition 4.7 for stability to hold.

First of all we try to determine $n^{out}(1)$. Notice that for stable coalitions $0 \leq n^{out}(N) \leq N-1$, which in our case ($N=1$) implies $n^{out}(1) = 0$. For this to hold, we need:

$$h(1,2) = \frac{1}{2\alpha c^2}\left(\frac{\gamma}{\alpha} + c^2\right) < \frac{1}{\alpha c}$$

that is:

$$\frac{\gamma}{\alpha} + (1-c)^2 < 1.$$

The conditions in Proposition 4.7 (i) are in this case only one: $h(0,1) > ph(1,2) + (1-p)h(0,2)$ where $h(0,1) = \frac{1}{\alpha c}$ and $h(0,2) = \frac{1}{2\alpha c^2}\left(1 + \frac{\gamma}{\alpha} + c^2\right)$. This condition is satisfied if and only if $\frac{\gamma}{\alpha} + (1-c)^2 < p$

Condition (ii) in Proposition 4.7 is empty because $n^{out} = 0$ and we have no inequalities to check. □

In Figure 3 we plot the boundary between stable and unstable 1-coalitions in the plane $\left(\frac{\gamma}{\alpha}, p\right)$, with c being fixed at 0.8. The non-cooperative populations are those having parameters in the white triangle and the cooperative populations have parameters in the grey area. The threshold π_1 is the line $\pi_1 = p(x) = x + (1-c)^2$ that separates the white from the grey area. The blue curve is given by the function $p^*_\infty(x) = \frac{1-c}{1-x}$ and represents the limiting boundary between stable and unstable coalitions as $N \to \infty$ (see also next subsection for more on this limiting curve). The area below the blue curve consists of coalitions that are optimally infinite. Consequently, the white areas that are situated below the blue curve are the strategically non-cooperative populations where the optimal decision consists in infinite cooperation. In other words, the Nash equilibria are suboptimal: individual entities that are not distressed choose to not cooperate and hence they achieve an expected lifetime of $\frac{1}{\alpha c}$, but infinite cooperation would lead to an infinite lifetime.

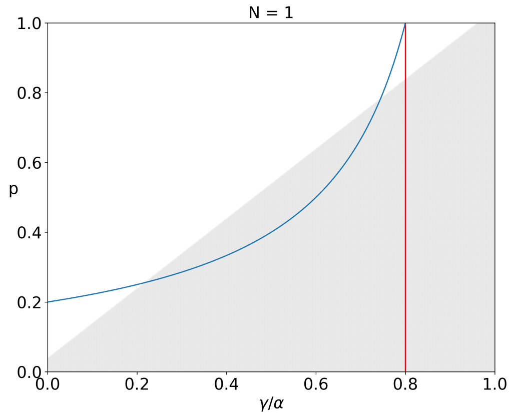

FIGURE 3. The repartition of the stable and unstable coalitions in the plane $(\frac{\gamma}{\alpha}, p)$ and for the sizes of the coalition $N = 1$. The vertical red line is the level of the parameter c that is fixed as $c = 0.8$. Grey area: unstable coalitions; white area: stable coalitions.

5.2. **Critical boundaries for stability and the inefficiency zones of the Nash equilibria.** As seen for the case $N = 1$, there is actually for any N a threshold probability $\pi_N \in [0,1]$ such that for $p > \pi_N$ the coalitions are stable and for $p \leq \pi_N$ the coalitions are unstable, when keeping the parameters c, $\frac{\gamma}{\alpha}$ fixed (see Appendix A.7 for a proof of this statement).

We now present a numerical study for stable and unstable regions in N-coalitions, using an algorithm based on Proposition 4.4.

In Figure 4, we analyse these regions in the $\left(\frac{\gamma}{\alpha}, p\right)$ plane for increasing values of N as follows (from up left, to right and downward) $N = 1$, $N = 2$, $N = 10$ $N = 20$ and $N = 100$. The parameters that are fixed for all pictures are $\alpha = 0.5$ and $c = 0.8$, while p is varying between 0 and 1 and γ is varying in the interval [0,0.5] so that $\frac{\gamma}{\alpha} \in [0, 1)$. As for the case $N = 1$, the white area represents the stable coalitions and the unstable coalitions are given in the grey area. The blue curve represents the boundary:

$$\pi := \frac{1-c}{1-\frac{\gamma}{\alpha}}$$

that was identified as the critical probability that separates optimal finite cooperation from optimal infinite cooperation (see Proposition 3.6). We will call π the *optimal threshold*, as it results from the optimal cooperation. The critical probabilities $(\pi_N), N \geq 1$ are given by the boundaries separating the white from the grey areas. We may observe that they converge as N increases to the asymptotic boundary π and at $N = 100$ it appears already very closed to it. In our numerical investigation, we could observe such a convergence of the sequence $(\pi_N), N \geq 1$ to the value π to occur often, but not systematically. The convergence towards π seems to break partially for large values of the parameter c, where we could sometimes observe segments of the boundary π_N remaining above π as N increases (see Appendix A.7 for a partial convergence result).

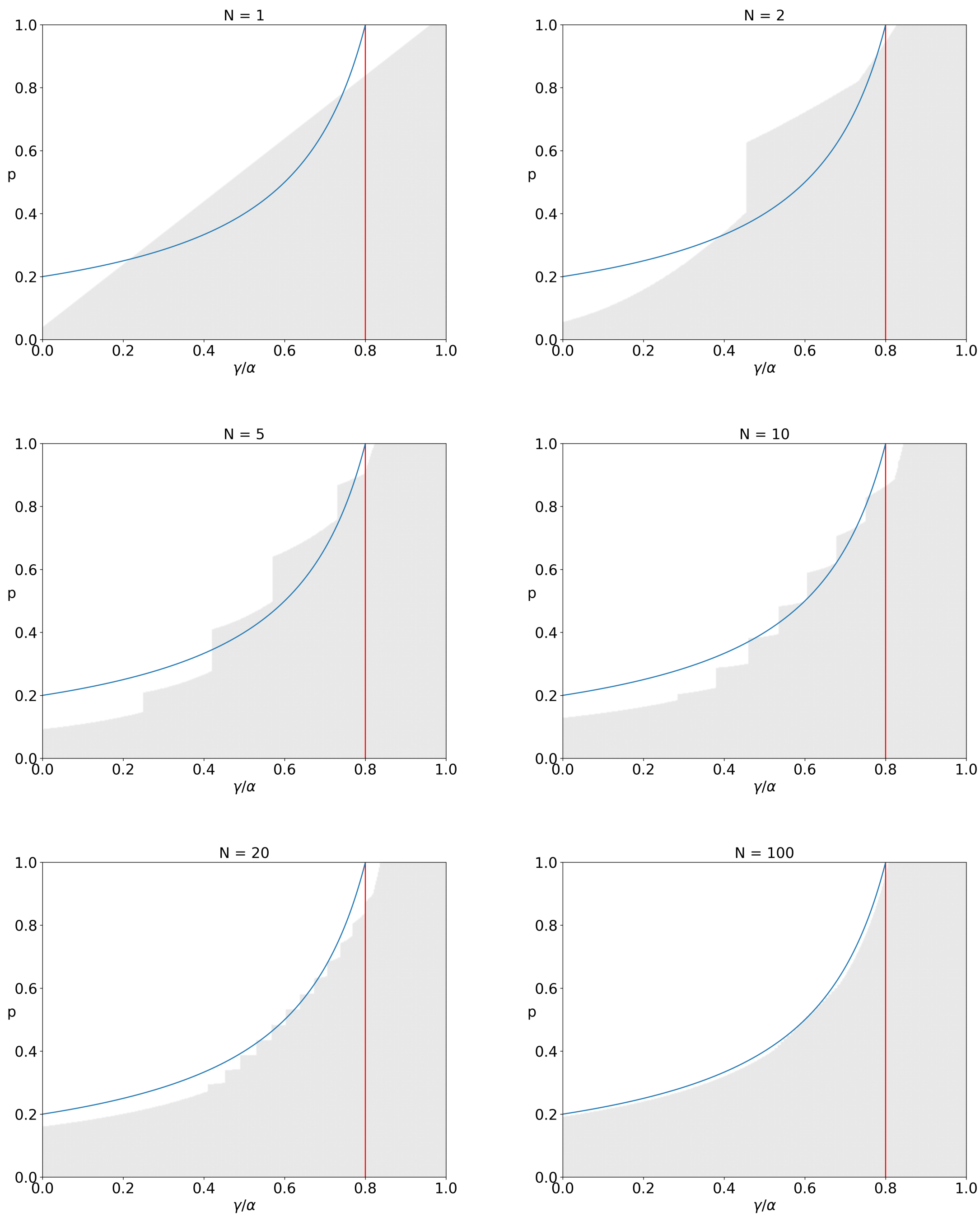

FIGURE 4. The repartition of the stable and unstable coalitions in the plane $(\frac{\gamma}{\alpha}, p)$ and for different sizes of the coalition N. The blue curve is the asymptotic frontier that delimitates the stable and unstable coalitions when $N \to \infty$, which is depending on p and $\frac{\gamma}{\alpha}$. The vertical red line is the level of the parameter c that is fixed as $c = 0.8$. Grey area: unstable coalitions; white area: stable coalitions.

An important element in the analysis of stable coalitions is to understand if they give rise to large inefficiencies as compared to the optimal coalitions. Optimal sizes of coalitions, denoted by N^* were introduced and studied in Subection 3.2. In order to measure possible inefficiencies, we introduce $N^{\min}$, as being the size of the smallest stable coalition, that is: if $N = N^{\min}$ an N-coalition is stable, and $N < N^{\min}$ implies an N-coalition is unstable. We set $N^{\min} = \infty$ whenever there are no stable coalitions. We shall name $N^{\min}$ *strategic coalition size*.

The size $N^{\min}$ appears to be the maximal size a group reaches within the strategic game $\Gamma = (\Gamma(N))_{N\geq 1}$, assuming it starts with a single healthy entity. If stable coalitions exist, then the game evolves through larger and larger coalitions until it reaches a size where it is stable and then, the game never ends. This is indeed $N^{\min}$. Larger size games $\Gamma(N)$ with $N > N_0$ will not be played. It can be the case that the game Γ stops before reaching the size $N^{\min}$, because of a smaller, unstable coalition becoming extinct, before getting the chance to enlarge.

In Figure 5, we display, for different values of the parameter c and in the plane $(\frac{\gamma}{\alpha}, c)$ the optimal N^* versus the strategic $N^{\min}$ sizes of coalitions. Next, in Figure 6 of expected lifetimes are measured, as corresponding to the optima coalition size N^* (left pictures) versus the strategic size $N^{\min}$ (right pictures).

Finally, we compute the differential in the expected lifetime of an optimal coalition as compared to the strategic size coalition. How far apart are these expected lifetimes? This gives the clearest picture of the inefficiency of the Nash equilibria. The answer is of course depending on the values of the parameters. Figure 7 shows the expected lifetime differential between the Pareto coalitions and the smallest stable coalitions (left pictures) and in relative value (left pictures), for different values of the parameters. We may observe that the higher inefficiencies are obtained around the optimal probability threshold π separating the optimally finite from the optimally infinite coalitions. Inefficiencies can be infinite (the black areas) when $N^{\min} < \infty$ while $N* = \infty$. Also, relatively high inefficiencies, that can reach 60-70% in loss of expected lifetime are to be observed for the strategically individualist populations, where actual optimal groups would be larger, but finite, as of 20 or 50 members. The relative inefficiencies for these strategically individualist populations, are increasing ase $\frac{\gamma}{\alpha}$ becomes close to c and the probability p increases to 1, meaning high viability ratio and high risk adverse selection.

A. Proofs

A.1. **Proof of Lemma 2.1.** We notice that $\bar{\boldsymbol{\mu}}$ corresponds to the conditional law of $\ell_0(k)$ under $\mathbb{P}^{(1)}$, given $\ell_0(k) \neq -1$, i.e.,

$$\bar{\boldsymbol{\mu}}(0) = \mathbb{P}^{(1)}[\ell_0(k) = 0|\ell_0(k) \neq -1] = c, \quad \bar{\boldsymbol{\mu}}(+1) = \mathbb{P}^{(1)}[\ell_0(k) = +1|\ell_0(k) \neq -1] = 1 - c.$$

Then, the equations in (3) write as:

$$\bar{\boldsymbol{\mu}}\bar{\mathbf{P}}(t) = \bar{\boldsymbol{\mu}}, \tag{20}$$

where $\bar{\mathbf{P}}(t) = (\bar{p}_{i,j}(t))_{i,j\in\{-1,0,+1\}}$ is the matrix of transition functions at time t of $\ell(k)$ given $\ell_t(k) \neq -1$. We observe that the equality above is precisely the definition of a stationary distribution. An equivalent expression of (20) is

$$\bar{\boldsymbol{\mu}}\bar{\mathbf{Q}} = 0, \tag{21}$$

where $\bar{\mathbf{Q}} = \lim_{t \downarrow 0} \frac{1}{t}(\bar{\mathbf{P}}_t - \bar{\mathbf{P}}_0)$, that is,

$$\bar{\mathbf{Q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\gamma & \gamma \\ 0 & \delta & -\delta \end{pmatrix}$$

We conclude that the solution of (21) is easily found to be $c = \frac{\delta}{\gamma+\delta}$.

A.2. **Proof of Lemma 2.6.** By definition, $\eta^-(\mathbf{L}_t^{(N)}) = 0$ means there are no distressed entities, that is:

$$\{\eta^-(\mathbf{L}_t^{(N)}) = 0\} = \{\mathbf{L}_t^{(N)} \in \{0, +1\}^N\}.$$

Obviously, if $A \notin \{0, +1\}^N$, then $\mathbb{P}^{(N)}\left(\mathbf{L}_t^{(N)} \in A \mid \eta^-(\mathbf{L}_t^{(N)}) = 0\right) = 0 = \bar{\boldsymbol{\mu}}^{(N)}(A)$ by definition of $\bar{\boldsymbol{\mu}}^{(N)}$.

Now, suppose $A \in \{0, +1\}^N$. We remark that $\{0, +1\}^N \subset I$. We consider t fixed, with $t \in [\theta_n, \theta_{n+1}]$ for some n, using the notation from Definition 2.5 . We write the event $\{\mathbf{L}_t^{(N)} \in \{0, +1\}^N\}$ as the union of two disjoint events:

$$B := \{\mathbf{L}_t^{(N)} \in \{0, +1\}^N\} = B_1 \cup B_2,$$

where $B_1 = \{\boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n} \in \{0, +1\}^N\}$ and $B_2 = \{\boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n} \notin I\} \cap \{\mathcal{R}(\boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n}) \in \{0, +1\}^N\}$. The result will follow from the analysis below:

- Suppose that $\boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n} \in I$. The law of $\boldsymbol{\ell}_t^{\mathbf{x}_n}$ conditionally on $\{\boldsymbol{\ell}_t^{\mathbf{x}_n} \in \{0, +1\}^N\}$ is $\bar{\boldsymbol{\mu}}^{(N)}$ by Lemma 2.1. Indeed, all components of $\boldsymbol{\ell}$ are independent and $\bar{\boldsymbol{\mu}}^{(N)}$ is the product measure $(\bar{\boldsymbol{\mu}})^N$. Conditionally on B_1, $\mathbf{L}_t^{(N)} = \boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n}$ therefore the law of $\mathbf{L}_t^{(N)}$ is $(\bar{\boldsymbol{\mu}})^N$, conditionally on B_1.
- Suppose that $\boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n} \notin I$. Then, $\mathbf{L}_t^{(N)} = \mathcal{R}(\boldsymbol{\ell}_{t-\theta_n}^{\mathbf{x}_n}) \in A$, by definition of the process $\mathbf{L}^{(N)}$. In this case, $t = \theta_{n+1}$. By Definition 2.4, for the process $\boldsymbol{\ell}$ to be reflected in a point belonging to $A \subset \{0, +1\}^N$, the reflection function necessarily is applied to the region E_+ so that we have all $\mathbf{L}_t^{(N)}(k)$ independent and identically distributed random variables Bernoulli with parameter $(1 - c)$. This is to say $\bar{\boldsymbol{\mu}}^{(N)}$ is the law of $\mathbf{L}_t^{(N)}$ conditionally on B_2.

Therefore:

$$\begin{aligned}
\mathbb{P}^{(N)}\left(\mathbf{L}_t^{(N)} \in A \mid B\right) =& \mathbb{P}^{(N)}\left(\mathbf{L}_t^{(N)} \in A | B_1\right) \cdot \mathbb{P}^{(N)}\left(B_1 | B\right) + \mathbb{P}^{(N)}\left(\mathbf{L}_t^{(N)} \in A | B_2\right) \cdot \mathbb{P}^{(N)}\left(B_2 | B\right) \\
=& \bar{\boldsymbol{\mu}}^{(N)}(A) \cdot \left(\mathbb{P}^{(N)}\left(A_1 | B\right) + \mathbb{P}^{(N)}\left(B_2 | B\right)\right) \\
=& \bar{\boldsymbol{\mu}}^{(N)}(A).
\end{aligned}$$

A.3. **Proof of Theorem 3.2.** For simplicity we use the notations $Y, \mathbb{E}, \mathbb{P}$ instead of $Y^{(N)}, \mathbb{E}^{(N)}, \mathbb{P}^{(N)}$ respectively for the distress process, the probability and expectation corresponding to the initial distribution $\boldsymbol{\mu}^{(N)}$.

We denote by $\{P_t, t \geq 0\}$ the semigroup associated with the transition matrix $\mathbf{Q}^{(N)}$. This means that P_t is characterised by an infinitesimal generator:

$$Ah(n) = \begin{cases} [\varphi(1) - \varphi(0)]\alpha N c^N & \text{for } n = 0, \\ [\varphi(n+1) - \varphi(n)]\alpha(N-n) + [\varphi(n-1) - \varphi(n)]\gamma(N-n) & \text{for } n = 1, .., N, \end{cases} \tag{22}$$

for any function φ bounded and continuous, defined as $\varphi : \{0, \cdots, N\} \to \mathbb{R}$. The semigroup P_t and its generator are linked by:

$$P_t\varphi - \varphi = \int_0^t P_s A\varphi ds. \tag{23}$$

We denote by $\mathcal{F}_t^Y := \sigma(Y_s, s \leq t)$, the information about the distress process up to time t. We need to prove that for any bounded continuous function φ we have:

$$\mathbb{E}\left[\varphi(Y_t)|\mathcal{F}_s^Y\right] = \mathbb{E}\left[\varphi(Y_t)|Y_s\right] = P_{t-s}\varphi(Y_s),$$

or, alternatively in view of the relation (23):

$$\mathbb{E}\left[\varphi(Y_t) - \varphi(Y_s)|\mathcal{F}_s^Y\right] = \int_s^t \mathbb{E}[\varphi(Y_u)|Y_s]\mathrm{d}u, \quad s \leq t. \tag{24}$$

where $\mathbb{E}_x$ denotes the expectation under $\mathbb{P}$ starting from x. We use the technique of localisation to show that the relation (24) holds for any $s \leq t$, see for eg. [12], page 36.

We consider the N-dimensional Markov chain with independent components $\boldsymbol{\ell} = (\ell(1), \cdots, \ell(N))$, as introduced in the previous section, i.e., $\boldsymbol{\ell}$ is the canonical realisation of the semigroup $P_t^{\boldsymbol{\ell}}$ characterised by the following infinitesimal generator:

$$A^{\boldsymbol{\ell}} f(\mathbf{x}) = \sum_{i:x(i)=0} \{[f(\mathbf{x}+\mathbf{e}_i) - f(\mathbf{x})]\gamma + [f(\mathbf{x}-\mathbf{e}_i) - f(\mathbf{x})]\alpha\} + \sum_{i:x(i)=+1} [f(\mathbf{x}-\mathbf{e}_i) - f(\mathbf{x})]\delta, \tag{25}$$

with $f : \{-1, 0, +1\}^N \to \mathbb{R}$, f continuous bounded, $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$, where 1 is the i component. The positive constants α, γ and δ are the ones from (1).

For each $\mathbf{x} \in \{-1, 0, +1\}^N$ let $\mathbb{P}_{\mathbf{x}}$ be such that $\boldsymbol{\ell}$ is a Markov process with initial distribution $\delta_{\mathbf{x}}$ (where $\delta_{\mathbf{x}}$ is the Dirac delta function at $\mathbf{x}$) and transition function $P_t^{\boldsymbol{\ell}}$. Also, we write $\mathbb{E}_{\mathbf{x}}$ for the expectation operator under $\mathbb{P}_{\mathbf{x}}$.

In what follows we use the notation introduced in Notation 2.2 and 2.3, and the additional notation:

$$\eta^0(\xi) := |\{i : x(i) = 0\}|,$$

so that $\eta^-(\mathbf{x}) + \eta^0(\mathbf{x}) + \eta^+(\mathbf{x}) = N$ for any $\mathbf{x} \in \{-1, 0, +1\}^N$.

Let us fix a state $\mathbf{x} \in I$; the neighbouring states (i.e., reachable from $\mathbf{x}$ after one transition of $\boldsymbol{\ell}$) are either in I or, when the process exits form I, at the boundary of E, defined as:

$$\partial E := \{\mathbf{x} \in \{-1, 0, +1\}^N \mid \min\{\eta^+(\mathbf{x}), \eta^-(\mathbf{x})\} = 1\}.$$

Indeed, the process $\boldsymbol{\ell}$ has independent components, so that it can evolve only by one component at a time. Additionally, we denote by

$$\partial E_+ := \partial E \cap E_+ \quad \text{and} \quad \partial E_- := \partial E \cap E_-.$$

We introduce the function:

$$g(\mathbf{x}) := \max\{\eta^-(\mathbf{x}) - \eta^+(\mathbf{x}), 0\}. \tag{26}$$

We notice that for $\mathbf{x} \in I$ the function g counts the number of distressed entities, that is:

$$g(\mathbf{x}) = \eta^-(\mathbf{x}) \text{ for all } \mathbf{x} \in I, \tag{27}$$

and at the boundary of E the function g removes one unit from the set of distressed entities:

$$g(\mathbf{x}) = \eta^-(\mathbf{x}) - 1 \text{ for all } \mathbf{x} \in \partial E. \tag{28}$$

For a fixed state $\mathbf{x} \in I$, let us detail the values taken by g for the neighbouring states:

(i) If $\eta^-(\mathbf{x}) = 0$ and $\eta^+(\mathbf{x}) = 0$ (hence $\eta^0(\mathbf{x}) = N$, i.e., $\mathbf{x} = \mathbf{0}$), then for $i = 1, \cdots, N$, the neighbouring states are:
$\mathbf{x} + \mathbf{e}_i \in I$ and $g(\mathbf{x} + \mathbf{e}_i) = \eta^-(\mathbf{x} + \mathbf{e}_i) = 0$,
$\mathbf{x} - \mathbf{e}_i \in I$ and $g(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x}) + 1 = 1$.

(ii) If $\eta^-(\mathbf{x}) = 0$ and $\eta^+(\mathbf{x}) > 0$, then the neighbouring states are:
$(ii.1)$ For $i \in \{k \mid x(k) = 0\}$:
$\mathbf{x} + \mathbf{e}_i \in I$ and $g(\mathbf{x} + \mathbf{e}_i) = \eta^-(\mathbf{x} + \mathbf{e}_i) = 0$,
$\mathbf{x} - \mathbf{e}_i \in \partial E_+$ and $g(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x} - \mathbf{e}_i) - 1 = \eta^-(\mathbf{x}) + 1 - 1 = 0$.
$(ii.2)$ For $i \in \{k \mid x(k) = +1\}$:
$\mathbf{x} - \mathbf{e}_i \in I$ and $g(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x}) = 0$.

(iii) If $\eta^-(\mathbf{x}) \in \{1, \cdots, N-1\}$, then (by definition of the set I) $\eta^+(\mathbf{x}) = 0$, the neighbouring states are:
$(iii.1)$ For $i \in \{k \mid x(k) = 0\}$ $\mathbf{x} + \mathbf{e}_i \in \partial E_+$ and $g(\mathbf{x} + \mathbf{e}_i) = \eta^-(\mathbf{x} + \mathbf{e}_i) - 1 = \eta^-(\mathbf{x}) - 1$,
$\mathbf{x} + \mathbf{e}_i \in \partial E_-$ and $g(\mathbf{x} + \mathbf{e}_i) = \eta^-(\mathbf{x} + \mathbf{e}_i) - 1 = \eta^-(\mathbf{x}) - 1$,
$\mathbf{x} - \mathbf{e}_i \in I$ and $g(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x} - \mathbf{e}_i) = \eta^-(\mathbf{x}) + 1$.
$(iii.2)$ For $i \in \{k \mid x(k) = -1\}$: no neighbouring states, as these states are absorbing.

(iv) If $\eta^-(\mathbf{x}) = N$, then $\mathbf{x} = (-1, \cdots, -1) \in I$, i.e., the absorbing state for $\boldsymbol{\ell}$, meaning that it has no neighbouring states.

Therefore, for $\mathbf{x} \in I$ we have:

$$\begin{aligned}
A^{\boldsymbol{\ell}}\varphi \circ g(\mathbf{x}) &= \mathbf{1}_{\{\eta^0(\mathbf{x})=N\}} \cdot [\varphi(1) - \varphi(0)]\, \alpha N \\
&\quad + \mathbf{1}_{\{\eta^-(\mathbf{x})>0\}} \cdot \eta^0(\mathbf{x}) \left\{ \left[\varphi(\eta^-(\mathbf{x}) - 1) - \varphi(\eta^-(\mathbf{x}))\right] \gamma + \left[\varphi(\eta^-(\mathbf{x}) + 1) - \varphi(\eta^-(\mathbf{x}))\right] \alpha \right\} \\
&= \mathbf{1}_{\{\mathbf{x}=\mathbf{0}\}} \, [\varphi(1) - \varphi(0)]\, \alpha N \qquad (29) \\
&\quad + \sum_{n=1}^{N} \mathbf{1}_{\{\eta^-(\mathbf{x})=n\}} \cdot (N-n) \left\{ [\varphi(n-1) - \varphi(n)]\, \gamma + [\varphi(n+1) - \varphi(n)]\, \alpha \right\}.
\end{aligned}$$

Above, we have used the fact that: if $\mathbf{x} \in I$ and $\eta^-(\mathbf{x}) = n > 0$, then $\eta^+(\mathbf{x}) = 0$ and hence $\eta^0(\mathbf{x}) = N - n$.

We define the first exit time of $\boldsymbol{\ell}$ from I as

$$T_1(\omega) := \inf\{t \geq 0 \mid \boldsymbol{\ell}_t(\omega) \notin I\},$$

and we introduce the shift operator $\tau(t), t \geq 0$ defined as:

$$\boldsymbol{\ell}_s(\tau(t,\omega)) = \boldsymbol{\ell}_{t+s}(\omega).$$

(see Revuz and Yor [21] page 36 for more details). It follows that $T_1 \circ \tau(s) = \inf\{t \geq s \mid \boldsymbol{\ell}_t(\omega) \notin I\}$. Also, we define recursively $T_n = T_1 \circ \tau(T_{n-1})$ for $n > 1$. It follows from the definition of Y and the relations (27)-(28) that the following relation holds:

$$Y_{t\wedge T_1} = g(\boldsymbol{\ell}_{t\wedge T_1}).$$

We first show the property (24) up to the first exit time from I, which is a stopping time for $(\mathcal{F}_t^{\boldsymbol{\ell}})$ (the filtration of the process $\boldsymbol{\ell}$), but it is not a stopping time in $(\mathcal{F}_t^Y)$, the filtration of the process Y. We consider φ as above and denote

$$f := \varphi \circ g,$$

with g as in (26). We obtain:

$$\begin{aligned}
&\mathbb{E}\left[\varphi(Y_{t\wedge(T_1\circ\tau(s))}) - \varphi(Y_s)|\mathcal{F}_s^Y\right] = \mathbb{E}\left[f(\boldsymbol{\ell}_{t\wedge(T_1\circ\tau(s))}) - f(\boldsymbol{\ell}_s)|\mathcal{F}_s^Y\right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[f(\boldsymbol{\ell}_{t\wedge(T_1\circ\tau(s))}) - f(\boldsymbol{\ell}_s)|\mathcal{F}_s^{\boldsymbol{\ell}}\right]|\mathcal{F}_s^Y\right] = \mathbb{E}\left[\mathbb{E}\left[\int_s^{t\wedge(T_1\circ\tau(s))} A^{\boldsymbol{\ell}} f(\boldsymbol{\ell}_u)du|\boldsymbol{\ell}_s\right] \mid \mathcal{F}_s^Y\right] \\
&\quad = \mathbb{E}\left[\int_s^t \mathbb{E}\left[\mathbf{1}_{\{T_1\circ\theta_s>u\}}\cdot A^{\ell} f(\boldsymbol{\ell}_u) \mid \mathcal{F}_u^Y\right] du \mid \mathcal{F}_s^Y\right].
\end{aligned}$$

We now compute the inner expectations above. If $T_1 > u$ then $\boldsymbol{\ell}_u \in I$. Consequently, using (27), we have

$$\mathbf{1}_{\{T_1>u\}} \cdot f(\boldsymbol{\ell}_u) = \mathbf{1}_{\{T_1>u\}} \cdot \varphi(\eta^-(\boldsymbol{\ell}_u)),$$

and using (29) we get the following.

Firstly, we suppose $Y_u = 0$, this is equivalent to $\eta^-(\boldsymbol{\ell}_u) = 0$. We obtain

$$\begin{aligned}
&\mathbf{1}_{\{Y_u=0\}}\cdot\mathbb{E}\left[\mathbf{1}_{\{T_1\circ\tau(s)>u\}}\cdot A^{\ell} f(\boldsymbol{\ell}_u) \mid \mathcal{F}_u^Y\right] = \mathbf{1}_{\{Y_u=0\}}\cdot\mathbb{E}\left[\mathbf{1}_{\{T_1\circ\tau(s)>u\}}\cdot\mathbf{1}_{\{\boldsymbol{\ell}_u=\mathbf{0}\}}\left[\varphi(1)-\varphi(0)\right]\alpha N \mid \mathcal{F}_u^Y\right] \\
&\qquad = \mathbf{1}_{\{Y_u=0\}}\cdot\left[\varphi(1)-\varphi(0)\right]\alpha N\cdot\mathbb{P}\left(T_1\circ\tau(s)>u, \boldsymbol{\ell}_u=\mathbf{0} \mid \mathcal{F}_u^Y\right) \\
&\qquad = \mathbf{1}_{\{Y_u=0\}}\cdot\left[\varphi(1)-\varphi(0)\right]\alpha N\cdot\mathbb{P}\left(\boldsymbol{\ell}_u=\mathbf{0} \mid \{T_1\circ\tau(s)>u\}\cap\mathcal{F}_u^Y\right)\cdot\mathbb{P}\left(T_1\circ\tau(s)>u \mid \mathcal{F}_u^Y\right) \\
&\qquad = \mathbf{1}_{\{Y_u=0\}}\cdot\left[\varphi(1)-\varphi(0)\right]\alpha N c^N\cdot\mathbb{P}\left(T_1\circ\tau(s)>u \mid \mathcal{F}_u^Y\right).
\end{aligned}$$

The first equality is obtained by using the observation that $\mathbf{1}_{\{\eta^-(\mathbf{x})=0\}} \cdot A^{\ell} h \circ g(\mathbf{x}) = 0$ for all $\mathbf{x}$ but $\mathbf{x} = \mathbf{0}$, where it equals $\varphi(1) - \varphi(0)$. The last equality uses the property that conditionally on $\eta^-(\boldsymbol{\ell}) = 0$, the process $\eta^+(\boldsymbol{\ell})$ is binomial with parameter $1 - c$ and $\{\boldsymbol{\ell}_u = \mathbf{0}\} = \{\eta^+(\boldsymbol{\ell}) = 0\}$.

Secondly, we suppose $Y_u > 0$. This implies that $n^-(\boldsymbol{\ell}_u) = Y_u$, $n^0(\boldsymbol{\ell}_u) = N - Y_u$, and $n^+(\boldsymbol{\ell}_u) = 0$. We get

$$\begin{aligned}
&\mathbf{1}_{\{Y_u>0\}}\cdot\mathbb{E}\left[\mathbf{1}_{\{T_1\circ\tau(s)>u\}} A^{\ell} f(\boldsymbol{\ell}_u) \mid \mathcal{F}_u^Y\right] \\
&= \mathbf{1}_{\{Y_u>0\}}\cdot\mathbb{P}\left(T_1\circ\tau(s)>u|\mathcal{F}_u^Y\right)(N-Y_u)\left\{\left[\varphi(Y_u-1)-\varphi(Y_u)\right]\gamma + \left[\varphi(Y_u+1)-\varphi(Y_u)\right]\alpha\right\}.
\end{aligned}$$

Putting together the two expressions, we have:

$$\mathbb{E}\left[\varphi(Y_{t\wedge(T_1\circ\tau(s))})-\varphi(Y_s)|\mathcal{F}_s^Y\right]=\mathbb{E}\left[\int_s^t\mathbb{P}\left(T^1\circ\tau(s)>u|\mathcal{F}_u^Y\right)\cdot A^\ell\varphi(Y_u)du\mid\mathcal{F}_s^Y\right].$$

Using the strong Markov property and the above result recursively, we can show that for any $n>0$ we have:

$$\mathbb{E}\left[\varphi(Y_{t\wedge T_n})-\varphi(Y_s)|\mathcal{F}_s^Y\right]=\mathbb{E}\left[\int_s^t\mathbb{P}\left(T_n>u|\mathcal{F}_u^Y\right)\cdot A^\ell\varphi(Y_u)du\mid\mathcal{F}_s^Y\right].$$

Taking the limit as n goes to infinity, using the dominated convergence theorem and the fact that there are finitely many interactions (i.e., times of exit from I) in any given interval $[s,t]$ we deduce that $\mathbb{P}(T^n>u|\mathcal{F}_u^Y)\to 1$ as $n\to\infty$. We conclude that indeed, equation (24) is verified for any s,t with $s\leq t$.

A.4. **Proof of Theorem 3.5.** As conditionally on the filtration $(\mathcal{F}_t^Y)$ all entities in the coalition have the same distribution, it follows that $\mathbb{P}\left(L_t^{(N)}(i)\in\{0,1\}\mid\mathcal{F}_t^Y\right)=\frac{N-Y_t^{(N)}}{N}$. Consequently:

$$\begin{aligned}h(n,N)&=\mathbb{E}\left[\int_0^\infty\mathbb{P}\left(L_t^{(N)}(i)\in\{0,1\}\mid\mathcal{F}_t^Y\right)dt\mid Y_0^{(N)}=n\right]\\&=\mathbb{E}\left[\int_0^\infty\frac{N-Y_t^{(N)}}{N}dt\mid Y_0^{(N)}=n\right].\end{aligned}$$

The functions $h(n,N),n\in\{0,...,N\}$ satisfy the following system of equations:

$$\begin{cases}\sum_{k=0}^N\left(h(k,N)-h(n,N)\right)q^{(N)}(n,k)=-\frac{N-n}{N}\\ h(N,N)=0,\end{cases}$$

that is:

$$\begin{cases}(h(1,N)-h(0,N))\alpha Nc^N=-1,\\(h(n+1,N)-h(n,N))\alpha(N-n)+(h(n-1,N)-h(n,N))\gamma(N-n)=-\frac{N-n}{N},\\h(N,N)=0.\end{cases}$$

Rearranging terms, we obtain:

$$\begin{cases}\alpha h(n+1,N)-(\alpha+\gamma)h(n,N)+\gamma h(n-1,N)+\frac{1}{N}=0,\\h(1,N)=h(0,N)-\frac{1}{N\alpha c^N},\\h(N,N)=0.\end{cases}\tag{30}$$

This is a second-order linear recursive equation. By defining $\Delta(n,N)=N(h(n,N)-h(n+1,N))$, we can transform the equation 30 into a first-order linear difference equation as:

$$\begin{cases}\Delta(n,N)=\frac{1}{\alpha}+\frac{\gamma}{\alpha}\Delta(n-1,N),\\\Delta(0,N)=\frac{1}{\alpha c^N},\end{cases}$$

with solution:

$$\Delta(n,N) = \begin{cases} \frac{1}{\alpha c^N} + \frac{n}{\alpha} & \text{, if } \gamma = \alpha, \\ \left(\frac{1}{\alpha c^N} - \frac{1}{\alpha-\gamma}\right)\left(\frac{\gamma}{\alpha}\right)^n + \frac{1}{\alpha-\gamma} & \text{, otherwise,} \end{cases} \tag{31}$$

for $n = 0, 1, \cdots, N$.

By the telescoping property of the difference sequence (and the fact that $h(N,N) = 0$), we have that

$$h(n,N) = h(n,N) - h(N,N) = \frac{1}{N}\sum_{k=n}^{N-1} \Delta(k,N), \tag{32}$$

and the result follows from equation (31).

Remark A.1. *An alternative representation of $h(n,N)$ obtained also from (32)-(31), which has the benefit of working for both $\alpha = \gamma$ and $\alpha \neq \gamma$, is:*

$$h(n,N) = \frac{1}{\alpha N c^N}\sum_{k=n}^{N-1}\left(\xi^k + c^N\sum_{j=0}^{k-1}\xi^j\right). \tag{33}$$

A.5. **Proof of Proposition 3.6.** Let $X \sim \mathrm{Bin}(p,N)$ so that $\mathbb{E}[X] = Np$. We observe that when $\xi = \frac{\gamma}{\alpha} = 1$, $\mathbb{E}[h(X,N)]$ diverges to $+\infty$ as $N \to +\infty$. We now study the case $\xi \neq 1$. Using Newton's binomial, we obtain:

$$\mathbb{E}\left[\xi^X\right] = \sum_{n=0}^{N}\xi^n\binom{N}{n}p^n(1-p)^{N-n} = (1-p+p\xi)^N$$

Using the representation of the function h in Theorem 3.5, we obtain:

$$\begin{aligned}\mathbb{E}[h(X,N)] &= \frac{1-\mathbb{E}[X]/N}{\alpha-\gamma} + \frac{1}{\alpha N c^N}\left(1 - \frac{c^N}{1-\xi}\right)\left(\frac{\mathbb{E}[\xi^X]-\xi^N}{1-\xi}\right) \\ &= h^*(p) + \left[\frac{1}{\alpha N c^N}\left(1-\frac{c^N}{1-\xi}\right)\left(\frac{(1-p+p\xi)^N - \xi^N}{1-\xi}\right)\right]\end{aligned}$$

If $\xi > 1$ the expression diverges to $+\infty$ as $N \to \infty$. If $\xi < 1$ this expression is not necessarily convergent since $c^N \to 0$ as $N \to \infty$ and it appears at the denominator. Nevertheless, ignoring terms that vanish asymptotically, we have that the term in the square brackets behaves as

$$\frac{1}{\alpha N c^N}\left(1-\frac{c^N}{1-\xi}\right)\left(\frac{(1-p+p\xi)^N-\xi^N}{1-\xi}\right) \sim \frac{1}{\alpha N(1-\xi)}\left(\frac{1-p+p\xi}{c}\right)^N$$

which converges (to zero) if and only if $c \geq 1-p+p\xi$. As $c, \xi \in (0,1)$, this last condition is equivalent to $\xi \leq c$ and $p \in [\frac{1-c}{1-\xi}, 1]$.

A.6. **Proof of Theorem 3.7.** As the entities interact, they are interdependent and the limiting process cannot be computed using the strong law of large numbers. We employ the theory presented in Ethier and Kurtz [4]).

We first write the process $Z_t^{(N)} := \frac{Y_t^{(N)}}{N}$ in the form of a stochastic differential equation (cf. Section 10.4 in [4]). So, for now, N is fixed. We define two counting processes $P_t^{+(N)}$ and $P_t^{-(N)}$. The process $P_t^{+(N)}$ counts the positive jumps of the process $Y_t^{(N)}$ and the process $P_t^{-(N)}$ counts the negative jumps of the process $Y_t^{(N)}$, so that:

$$Y_t^{(N)} = Y_0^{(N)} + P_t^{+(N)} - P_t^{-(N)}.$$

It follows from the transition matrix of the process $Y^{(N)}$ that is given in Theorem 3.2, that the predictable compensator of the process $P_t^{+(N)}$ is given by process

$$\int_0^t q^{(N)}((Y_s^{(N)}), i+1)\mathbf{1}_{\{Y_s^{(N)}<N\}} ds = \int_0^t \alpha(N - Y_s^{(N)}) ds$$

is and the predictable compensator of the process $P_t^{-(N)}$ is the process

$$\int_0^t q^{(N)}((Y_s^{(N)}), i-1)\mathbf{1}_{\{Y_s^{(N)}>0\}} ds = \int_0^t \gamma(N - Y_s^{(N)})\mathbf{1}_{\{Y_s^{(N)}>0\}} ds$$

Therefore, the following processes are martingales:

$$M_t^{+(N)} = P_t^{+(N)} - \int_0^t \beta^{+(N)}(Y_s^{(N)}) ds, \text{ and } M_t^{-(N)} \quad = P_t^{-(N)} - \int_0^t \beta^{-(N)}(Y_s^{(N)}) ds, \quad (34)$$

where $\beta^{+(N)}(i) = q^{(N)}(i, i+1), \quad \beta^{-(N)}(i) = q^{(N)}(i, i-1)\mathbf{1}_{\{i>0\}}$.

We denote by $M_t^{(N)} = M_t^{+(N)} - M_t^{-(N)}$ and $\beta^{(N)} = \beta^{+(N)} - \beta^{-(N)}$ and we obtain

$$M_t^{(N)} = \left(P_t^{+(N)} - P_t^{-(N)}\right) - \int_0^t \beta^{(N)}(Y_s^{(N)}) ds$$

is also a martingale. Therefore $Y_t^{(N)} = Y_0^{(N)} + M_t^{(N)} - \int_0^t \beta^{(N)}(Y_s^{(N)}) ds$ and $Z^{(N)}$ satisfies the following SDE:

$$Z_t^{(N)} = Z_0^{(N)} + \frac{M_t^{(N)}}{N} - \int_0^t \frac{\beta^{(N)}(N Z_s^{(N)})}{N} ds. \quad (35)$$

We want to calculate the limit of the process $Z^{(N)}$, when N tends to infinity, therefore we compute the limit of each term in the left hand side of (35). The martingale $\frac{M_t^{(N)}}{N}$ is a finite variation martingale bounded, because it takes values in [0,1]), and with jumps of size $\pm\frac{1}{N}$. We have that the relation (10.1) in the Theorem 10.5 from [15] is satisfied:

$$\frac{1}{N} \sup_{s \le t} |M_s^{(N)} - M_{s-}^{(N)}| \le \frac{1}{N} \underset{N\to\infty}{\longrightarrow} 0.$$

The quadratic variation of $\frac{M_t^{(N)}}{N}$ is the sum of the square of its jumps:

$$[M^{(N)}]_t/(N^2) = \frac{1}{N}\left(\frac{P_t^{+(N)}}{N} + \frac{P_t^{-(N)}}{N}\right).$$

We have $\beta^{+(N)}(Y_t^{(N)}) \leq \alpha N$ and $\beta^{-(N)}(Y_t^{(N)}) \leq \gamma N$, therefore using (34), we obtain:

$$\mathbb{E}\left[\frac{P_t^{+(N)}}{N}\right] = \mathbb{E}\left[\frac{\int_0^t \beta^{+(N)}(Y_s^{(N)})ds}{N}\right] \leq \frac{\alpha t N}{N} = \alpha t,$$

$$\mathbb{E}\left[\frac{P_t^{-(N)}}{N}\right] = \mathbb{E}\left[\frac{\int_0^t \beta^{-(N)}(Y_s^{(N)})ds}{N}\right] \leq \frac{\gamma t N}{N} = \gamma t,$$

and:

$$\mathbb{E}\left[M^{(N)}]_t/(N^2)\right] \underset{N\to\infty}{\longrightarrow} 0.$$

Hence, from Theorem 10.5 and Remark 10.6 from [15] (see also Theorem 7.1.4 in [4]) we have that $\frac{M_t^{(N)}}{N} \to 0$, when N tends to infinity.

It remains to compute the limit when N tends to infinity for $\int_0^t \frac{\beta^{(N)}(NZ_s^{(N)})}{N}ds$, the third term in the expression (35) of $Z^{(N)}$. We write the transition intensities as functions of $z := \frac{i}{N}$ and N using the Theorem 3.2, we have:

$$\begin{aligned}
\frac{\beta^{+(N)}(Nz)}{N} &= \frac{q^{(N)}(i,i+1)(i)}{N} = \alpha\frac{c^N}{N} \underset{N\to\infty}{\longrightarrow} 0, \quad \text{if } z = 0,\\
\frac{\beta^{+(N)}(Nz)}{N} &= \frac{q^{(N)}(i,i+1)(i)}{N} = \alpha(1-z), \quad \text{if } z \in \left\{\frac{1}{N}, \cdots, \frac{N-1}{N}, 1\right\},\\
\frac{\beta^{-(N)}(Nz)}{N} &= \frac{q^{(N)}(i,i-1)(i)}{N} = \gamma(1-z), \quad \text{if } z \in \left\{\frac{1}{N}, \cdots, \frac{N-1}{N}, 1\right\},\\
\frac{\beta^{-(N)}(Nz)}{N} &= 0, \quad \text{if } z = 0.
\end{aligned}$$

It follows that:

$$\int_0^t \frac{\beta^{(N)}(NZ_s^{(N)})}{N}ds = \int_0^t \frac{(\beta^{+(N)} + \beta^{-(N)})(NZ_s^{(N)})}{N}ds \underset{N\to\infty}{\to} \int_0^{t\wedge\tau} (1-Z_s)(\alpha-\gamma)ds,$$

where $\tau = \inf\{t \mid Z_t = 0\}$. Finally, supposing $z_0 = \lim_{N\to\infty} Y_0^{(N)}/N$ exists, we obtain $Z_t := \lim_{N\to\infty} Z_t^{(N)}$ is given by:

$$Z_t = z_0 - \int_0^{t\wedge\tau} (1-Z_s)(\alpha-\gamma)ds,$$

and the result follows.

A.7. **Proof of the existence of critical probabilities and convergence results.** This appendix brings some clarification about the existence of the critical probabilities π_N and their the convergence to π in some cases. First, we need to show that:

Lemma A.2. *The mapping $n \mapsto h(n, N)$ is strictly decreasing.*

Proof. [Proof of the Lemma] From (31) and (32) we obtain that for any N

$$h(n, N) - h(n+1, N) = \frac{\Delta(n, N)}{N} > 0.$$

Indeed, in the case $\alpha \neq \gamma$, $\Delta(n, N) = \frac{1}{\alpha c^N} \left(\frac{\gamma}{\alpha}\right)^n + \frac{1}{\alpha - \gamma} \left(1 - \left(\frac{\gamma}{\alpha}\right)^n\right)$ and this is nonnegative, as both terms of the sum are nonnegative. □

Using Lemma A.2, the conditions in Proposition 4.7 (i) can be written in terms of inequalities for p and the conditions for stability in Proposition 4.7 are:

$$(i) \quad p > \frac{h(n, N+1) - h(n, N)}{h(n, N+1) - h(n+1, N+1)}, \quad \text{for } n \in \{0, \ldots, n^{out}(N)\} \tag{36}$$

$$(ii) \quad h(n, N) - h(n+1, N+1) > 0, \quad \text{for } n \in \{n^{out}(N)+1, \ldots, N-1\}. \tag{37}$$

Suppose that the parameters α, γ, δ are fixed and are such that $\frac{\gamma}{\alpha} < c$. We have that $n^{out}(N) > 0$ for all N (as $h(0, N+1) > h(0, 1)$) so that in (36) we aways have a non empty set of conditions to verify.

Suppose that either there is $n \in \{0, ..., n^{out}(N)\}$ so that $\frac{h(n,N+1)-h(n,N)}{h(n,N+1)-h(n+1,N+1)} \geq 1$, or there is $n \in \{n^{out}(N)+1, \ldots, N-1\}$ with $h(n, N) - h(n+1, N+1) \leq 0$. Then, the statement of the proposition is true with $\pi_N = 1$, and reads: there are no stable N-coalitions.

Now, we suppose that there exist stable N-coalitions. Then, it means that the conditions in (37) are fulfilled and $\frac{h(n,N+1)-h(n,N)}{h(n,N+1)-h(n+1,N+1)} < 1$ for all $n \in \{0, ..., n^{out}(N)\}$. We denote

$$\pi_N = \max_{n \in \{0, \ldots, n^{out}(N)\}} \frac{h(n, N+1) - h(n, N)}{h(n, N+1) - h(n+1, N+1)}.$$

For any $p > \pi_N$ the conditions in (36) are verified and the corresponding N-coalitions are stable.

We now claim that when $n^{out}(N)/N \to 0$, then $\pi_N \to \pi$. For analysing the limit behavior of π_N we shall use the expression (11) of h:

$$h(n, N) = h^*(n/N) + \frac{1}{\alpha N c^N} \left(1 - \frac{c^N}{1-\xi}\right) \frac{\xi^n - \xi^N}{1-\xi}. \tag{38}$$

Notice that the first summand converges to the constant value $h^*(0) = \frac{1}{\alpha - \gamma}$, while the second summand diverges to $+\infty$ as $N \to \infty$.

We now show that when N sufficiently large, the conditions (37) are fulfilled. In order to study the sign it is convenient to rescale the sequence by $\alpha N c^N$, notice that this does not change its sign:

$$\lim_{N \to \infty} \alpha N c^N \left(h(n, N) - h(n+1, N+1)\right) = \lim_{N \to \infty} \frac{\xi^n}{1-\xi} - \frac{N}{N+1} \frac{1}{c} \frac{\xi^{n+1}}{1-\xi} = \frac{\xi^n}{1-\xi} \left(1 - \frac{\xi}{c}\right)$$

which is strictly positive if $\xi < c$ and independently of n, which proves the earlier claim.

This means that when N sufficiently large, only the conditions in (36) need to be fulfilled. Assuming that $n^{out}(N)/N \to 0$, it follows that it is sufficient to verify (36) for n as it becomes negligible for N large. We obtain:

$$\lim_{N\to\infty} h(n,N) = h^*(0) + \lim_{N\to\infty} \frac{1}{\alpha N c^N} \frac{\xi^n - \xi^N}{1-\xi} + \frac{\xi^n - \xi^N}{\alpha N(1-\xi)^2}.$$

Finally:

$$\lim_{N\to\infty} \frac{h(n,N+1) - h(n,N)}{h(n,N+1) - h(n+1,N+1)} = \lim_{N\to\infty} \frac{\frac{1}{\alpha(N+1)c^{N+1}} \frac{\xi^n}{1-\xi} - \frac{1}{\alpha N c^N} \frac{\xi^n}{1-\xi}}{\frac{1}{\alpha(N+1)c^{N+1}} \left(\xi^n + c^N \sum_{j=0}^{n-1} \xi^j\right)}$$

$$= \lim_{N\to\infty} \frac{1}{1-\xi} \left(1 - \frac{N+1}{N} c\right) = \frac{1-c}{1-\xi}.$$

Therefore we must require $p > \frac{1-c}{1-\xi} \in (0,1)$ if we want the conditions to be satisfied. Notice that the limit is independent of n. We can conclude that in the large N limit satisfying (36) for any n (say, $n = 0$) is equivalent to satisfying it for all n. Therefore for stability in the large N limit, only the first set of conditions (36) matter and stability is equivalent to ask $p > \pi$, as claimed.

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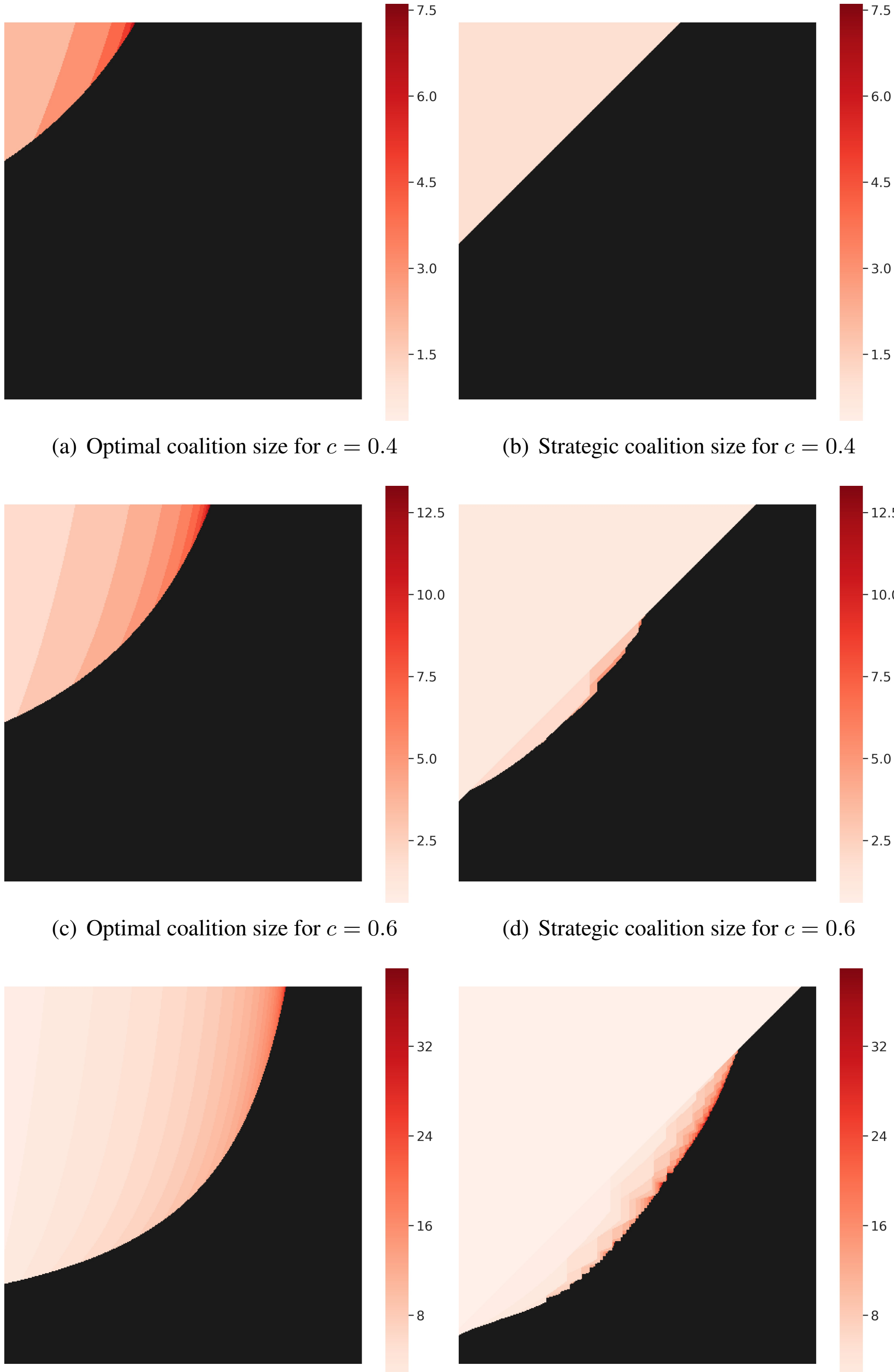


(a) Optimal coalition size for $c = 0.4$

(b) Strategic coalition size for $c = 0.4$

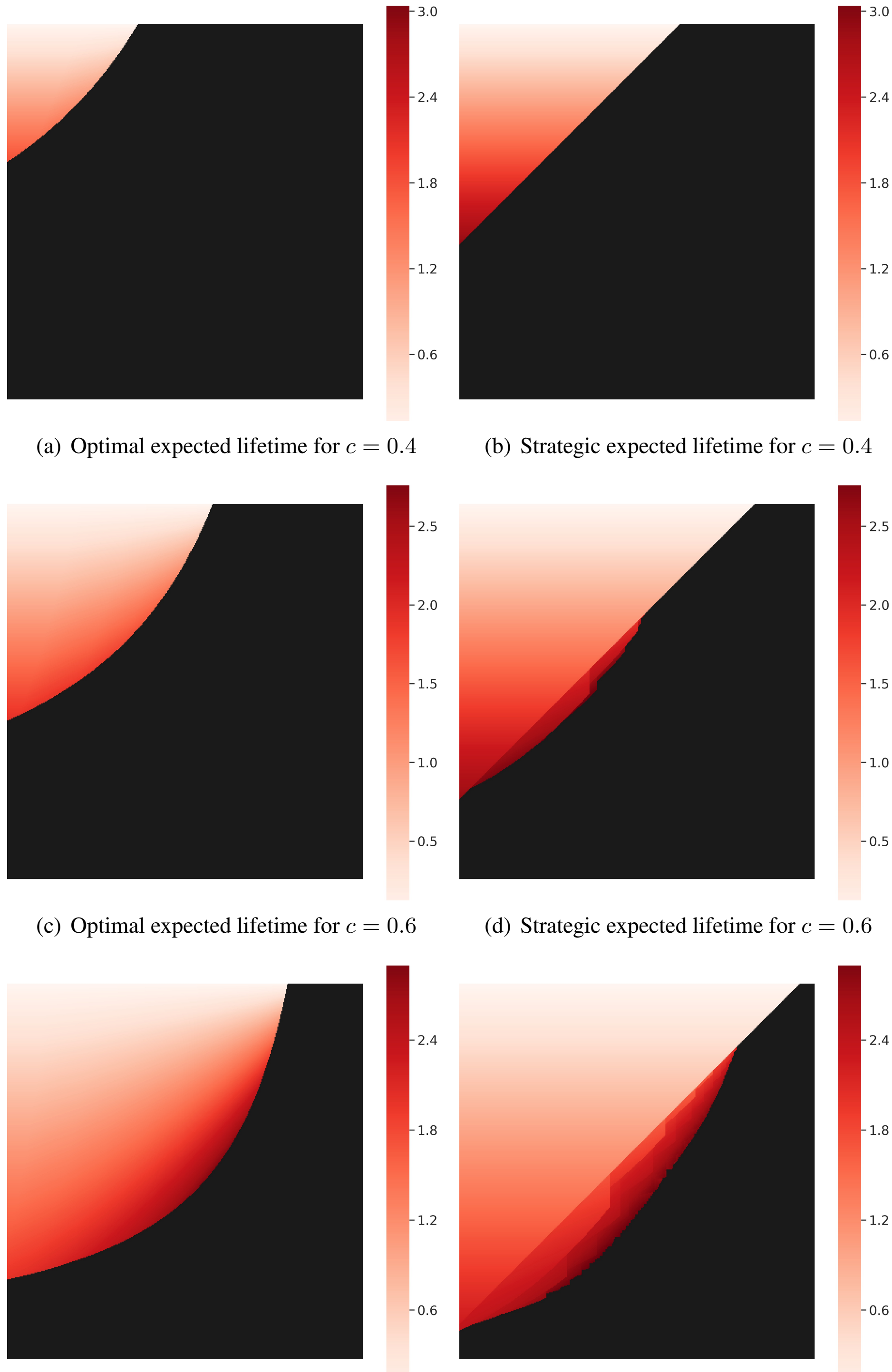
(c) Optimal coalition size for $c = 0.6$

(d) Strategic coalition size for $c = 0.6$

(e) Optimal coalition size for $c = 0.8$

(f) Strategic coalition size for $c = 0.8$

FIGURE 5. Optimal, versus strategic sizes of coalitions in the plane $(\frac{\gamma}{\alpha}, p)$ and for different values of the parameter c. The black colour represents infinite sizes, the other colours are in the legend



(a) Optimal expected lifetime for $c = 0.4$

(b) Strategic expected lifetime for $c = 0.4$

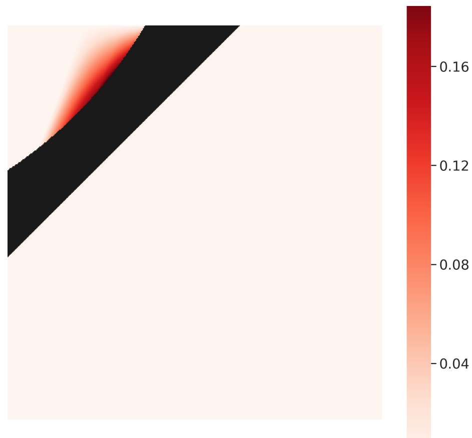
(c) Optimal expected lifetime for $c = 0.6$

(d) Strategic expected lifetime for $c = 0.6$

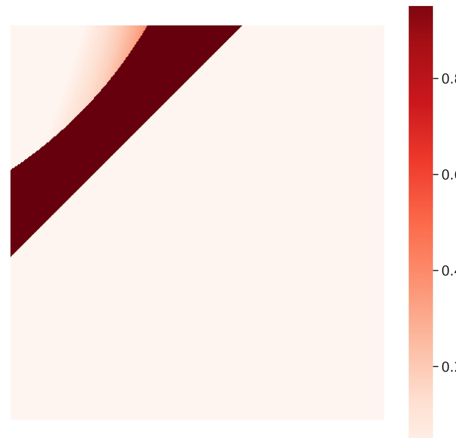
(e) Optimal expected lifetime for $c = 0.8$

(f) Strategic expected lifetime for $c = 0.8$

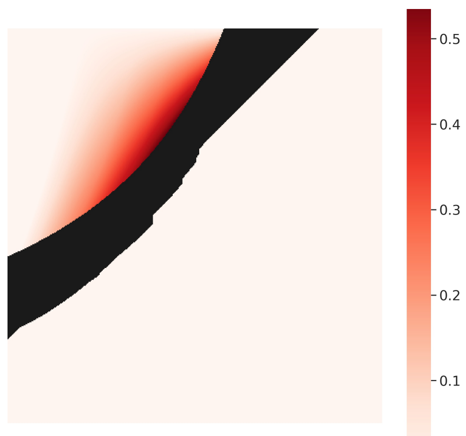
FIGURE 6. Expected lifetimes corresponding to optimal coalition sizes N^*, versus strategic coalition sizes $N^{\min}$ in the plane $(\frac{\gamma}{\alpha}, p)$ and for different values of the parameter c. The black colour represents infinite lifetimes, the other colours are in the legend.



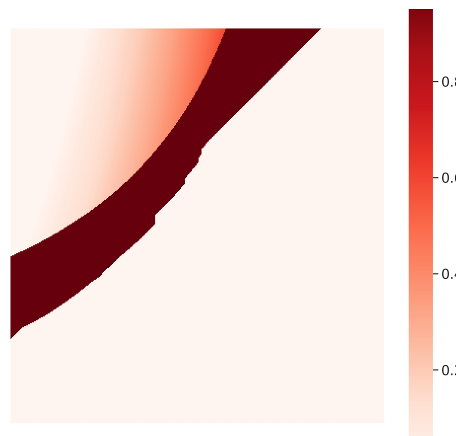
(a) Absolute loss in expected lifetime, $c = 0.4$.



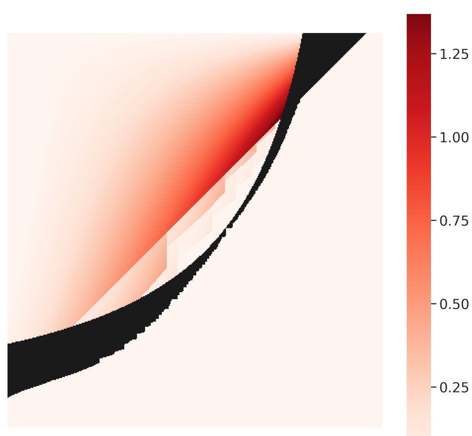
(b) Relative loss in expected lifetime, $c = 0.4$.



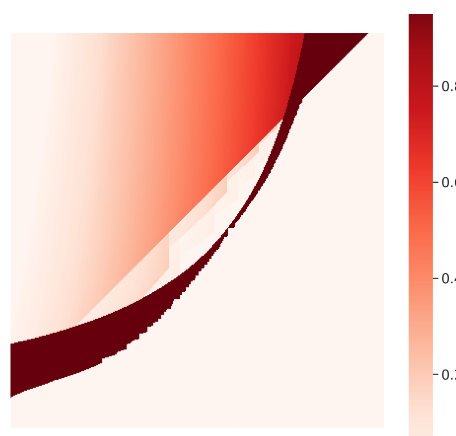
(c) Absolute loss in expected lifetime, $c = 0.6$.



(d) Relative loss in expected lifetime, $c = 0.6$.



(e) Absolute loss in expected lifetime, $c = 0.8$.



(f) Relative loss in expected lifetime, $c = 0.8$.

FIGURE 7. Absolute, versus relative losses that occur when a coalition has the strategic size $N^{\min}$ as compared to the optimal size N^*. Losses are expressed in terms of expected lifetime differential; they are displayed in the plane $(\frac{\gamma}{\alpha}, p)$ and for different values of the parameter c, as indicated below each subfigure. The black colour represents infinite absolute losses; we normalise these to 1 to obtain relative losses. The other relative losses are given by absolute losses divided by expected lifetime for optimal coalitions.