Option Pricing Under Stochastic Volatility

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Carr and Dupire on Volatility
Fischer Black once wrote:

Suppose we use the standard deviation ... of possible future returns on a stock ... as a measure of its volatility. Is it reasonable to take that volatility as constant over time? I think not.

- In this session, I provide a selective survey of some models for valuing European options on a single underlying which has stochastic volatility.

- There are now several books on this subject, so I apologize in advance to authors whose work I miss.

- There are two kinds of models covered in my talk:
  1. Models that are Markovian in Stock Price and Time
Models that are Markovian in Stock Price and Time

- We review ≤ 4 approaches for pricing options in a Markovian framework:
  1. Path Independent Diffusions: Assumes that stock price is a function of a contemporaneous tractable diffusion and time, possibly only after a path-independent measure change and deterministic time change.
  2. Local vol: a generalization of the above, which maintains price continuity, but only requires that instantaneous volatility be a function of stock price and time.
  3. Lévy processes: stock returns have stationary independent increments.
  4. Local Lévy: a generalization of the above, where the jump compensator is the product of the local aggregate arrival rate and a parametrically specified arrival rate specific to the jump size.

- This list is not exhaustive! For example, combinations of the above are allowed.
Standard Assumptions

- Unless indicated otherwise, we will always assume:
  - frictionless markets, eg. no transactions costs, no b/a spreads
  - no arbitrage (and hence the existence of an EMM $\mathbb{Q}$)
  - constant dividend yield $q$ (often 0)
  - constant riskfree rate $r$ (often 0)
  - a continuous time stochastic process for the stock price $S$. 
Approach #1: Path-Independent Diffusions

- In this class of models, we specify a tractable diffusion as a driver and then we assume that the underlying stock price is a function of the driver and time, perhaps only after picking a convenient numeraire and clock.

- The only diffusion drivers that we will cover are standard Brownian motion (SBM), and a Bessel process.

- For example, the Black Scholes model assumes that the stock price $S$ is the following exponential function of a standard Brownian motion $B$ and time $t$:

  $$S_t = S_0 e^{\mu t} e^{\sigma B_t - \frac{\sigma^2 t}{2}}, \quad t \geq 0,$$

  where $S_0 > 0$ is the initial stock price, $\mu \in \mathbb{R}$ is the required relative drift and $\sigma \in \mathbb{R}$ is the assumed constant (lognormal instantaneous) volatility.
Other Models with a Brownian Driver

- Beating Black Scholes by $\geq 70$ years, Bachelier described the stock price $S$ by a different function of a standard Brownian motion $B$ and time $t$:

$$S_t = e^{\mu t} [S_0 + aB_{\tau(t)}], \quad t \geq 0,$$

where $S_0 > 0$ is the initial stock price, $\mu$ is the required relative drift, $a$ is the assumed constant (normal instantaneous) volatility, and $\tau(t) \equiv \frac{1-e^{-2\mu t}}{2\mu}$ is a deterministic time change.

- Taking the total derivative and using the Brownian scaling property leads to the following dynamics:

$$dS_t = \mu dt + a dW_t, \quad t \geq 0,$$

where $W$ is a standard Brownian motion.
Bessel Process

- A Bessel process is a univariate diffusion $R$ with positive initial level $R_0$ and which uniquely solves the following stochastic differential equation:

\[ dR_t = \frac{2\nu + 1}{2R_t} dt + dB_t, \quad t > 0, \]

where $B$ is an SBM and $\nu \in \mathbb{R}$ is called the index of the Bessel process.

- The Bessel process is always positive and clearly not a martingale.

- Suppose $\nu = \frac{\delta}{2} - 1$ where $\delta$ is a positive integer. Then the Bessel process $R$ is the radial distance to the origin of the sum of $R_0 > 0$ and a $\delta$–dimensional standard standard Brownian motion.

- A possible reason for the name “Bessel Process” is that the modified Bessel function $I_\nu(\cdot)$ appears in the closed form solution for the transition probability density function. This also explains the non-intuitive parametrization of the drift term in the SDE in terms of $\nu$. 
Constant Elasticity of Variance (CEV) process

- From an SDE perspective, the Black Scholes model assumes that the stock price $S$ follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t > 0,$$

while the Bachelier model instead assumes an OU process:

$$dS_t = \mu S_t dt + \sigma dB_t, \quad t > 0.$$

- In an effort to generalize both processes while introducing a skewness parameter consistent with a lower bound of zero on the stock price, Cox (1975) introduced the following CEV process:

$$dS_t = \mu S_t dt + \sigma S_t^p dB_t, \quad t > 0, \text{ where } p \leq 1.$$
CEV as a Function of a Bessel Driver

- Recall that a Bessel process $R$ with index $\nu \in \mathbb{R}$ solves the following SDE:
  \[ dR_t = \frac{2\nu + 1}{2R_t} dt + dB_t, \quad t > 0, \]
  where $R_0 > 0$ and $B$ is an SBM, while the CEV process instead solves:
  \[ dS_t = \mu S_t dt + \sigma S_t^p dB_t, \quad t > 0, \text{ where } S_0 > 0 \text{ and } p \leq 1. \]

- As shown in Delbaen and Shirakawa (2002) and Cox (1975), if we let:
  \[ \hat{S}_t = e^{\mu t} R_{\tau(t)}^{-2\nu}, \text{ where } \tau(t) \equiv \frac{\sigma^2 e^{-\frac{1-\nu}{2\nu} \mu t} - 1}{2\nu (1-\nu) \mu}, \]
  then from Itô’s formula and the Brownian scaling property:
  \[ d\hat{S}_t = \mu \hat{S}_t dt - \sigma \hat{S}_t^{\frac{\nu+1}{2\nu}} dW_t, \]
  where $W$ is an SBM.

- Hence, if we set $p \equiv \frac{\nu+1}{2\nu}$, then $\hat{S}$ has the same law as the solution to the SDE for the CEV process.
Models with Affine Drift

- So far all of the models have required that the stock price have constant proportional drift.
- A simple way to let the stock price drift be affine instead of proportional is to let the cushion $C_t \equiv S_t - F$ be a drifting geometric Brownian motion:
  \[ C_t = C_0 e^{(\mu-\sigma^2/2)t + \sigma B_t}, \quad t > 0, \]
  where $B$ is a standard Brownian motion.
- Using Itô’s formula, the stock price’s drift and volatility are both affine:
  \[ dS_t = \mu(S_t - F)dt + \sigma(S_t - F)dW_t, \quad t > 0. \]
- $S_t$ can be interpreted as the future value at $t$ in a zero interest rate economy when an initial investment of $S_0$ is split between $F$ in a riskless asset and $C_0 \equiv S_0 - F$ in a drifting GBM.
- Constant Proportional Portfolio Insurance (CPPI) is the special case when $S$ is interpreted as wealth and $F < S_0$ is interpreted as the floor. Since a GBM is always positive, the price process $S$ is always above $F$. 
Affine Drift and Bessel Drivers

• Recall that for the dynamics on the last slide, the stock price is bounded below by the floor $F$. Setting $F = 0$ forces the stock price drift to be proportional. As a result, constant dollar dividends cannot be considered.

• In his Nobel prize winning paper, Merton (1973) considered the process:

$$dS_t = (\mu S_t - c)dt + \sigma S_t dB_t, \quad t > 0,$$

where $S_0 > 0$, and where $B$ is an SBM. Here, $c \geq 0$ is a constant dividend paid continuously over time until the $S$ process first hits zero.

• Merton derived the solution for a perpetual option on such a process, while Lewis (1997) derived the solution for the finite-lived case.

• Similarly, in their path-breaking paper, Cox Ross (1976) suggested the SDE:

$$dS_t = (\mu S_t - c)dt + \sigma \sqrt{S_t} dB_t, \quad t > 0.$$

• In both cases, the key is to be able to write the $S$ process as a function of a Bessel process, running on a deterministically time-changed clock.
Carr, Lipton, & Madan (2000) assume that the risk-neutral process for a state variable $S$ is:

$$dS_t = b(S_t, t)dt + a(S_t, t)dW_t, \quad t \in [0, T],$$

and that the discount rate is level-dependent $r_t = c(S_t, t)$.

In this general setup, the values of path-independent equity options or interest rate derivatives solve a PDE with variable coefficients.

Under weak restrictions on the arbitrary coefficients $\{a(S, t), b(S, t), c(S, t)\}$, a scale change of $S$ to $X = f(S, t)$ can render unit vol. Expressing the price of the overlying (ONLY) in terms of a new (dividend-paying) numeraire can also render zero drift (and keep vol at one). We thus obtain a normalized PDE with one coefficient (called either the killing rate or potential).
A Further Generalization (con’d)

• If this killing rate/potential is quadratic in $X$, then it can be eliminated through deterministic time change, along with further changes in the underlying and overlying, leaving the (backward) heat equation (see Carr, Lipton, Madan 2000).

• Alternatively, if this discount rate is a quadratic in $X$ plus a nonzero function of time divided by $X^2$, then we can reduce to a Bessel PDE (see Linetsky’s IJTAJ spectral representation paper for details).

• Recently, Carr, Laurence, and Wang (2006) derive an expression which the normal volatility coefficient must satisfy in order that a driftless diffusion without killing/discounting can be reduced to a Bessel process.
References


Approach #2: Local Vol

- Define a complete implied volatility surface as a $C^{2,1}$ function whose output is the Black Scholes implied volatility and whose two inputs are some measure of moneyness, e.g. strike $K$ and some measure of time to maturity (e.g. calendar time to maturity $T$).

- In the non-parametric approach, one first uses splines or regressions or an arbitrage-free option pricing model to obtain a complete implied volatility surface as in ad hoc BS. One can then convert these implieds to call prices $C(K, T)$.

- Dupire (1994) showed that under no arbitrage and the assumption that:

$$dS_t = \mu_t S_t dt + a(S_t, t)dW_t, \quad t \geq 0,$$

the deterministic function $a(S, t)$ can be inferred from the term and strike structure of call prices and hence implied vol’s.
Local Vol (Con’d)

• At a given time $t \geq 0$ and for a fixed spot price level $K$ and a fixed point in calendar time $T \geq t$, define $a_t^2(K, T)$ as the local variance at $(K, T)$:

$$a_t^2(K, T) \equiv \lim_{\Delta t \downarrow 0} \mathbb{E} \left\{ \frac{(S_{T+\Delta t} - S_T)^2}{\Delta t} \mid S_T = K, \mathcal{F}_t \right\},$$

where $\mathcal{F}_t$ is the filtration (information set) at $t$.

• In general, the local variance $a_t^2(K, T)$ is a nonnegative scalar stochastic process for each fixed $(K, T)$.

• However, Dupire (1994) assumed that the filtration just consists of stock prices and that the stock price is Markovian in itself and time. As a result, local variance is a constant scalar process for each fixed $(K, T)$:

$$a_t^2(K, T) = \lim_{\Delta t \downarrow 0} \mathbb{E} \left\{ \frac{(S_{T+\Delta t} - S_T)^2}{\Delta t} \mid S_T = K \right\} \equiv a^2(K, T).$$

• The square root of the function on the right is being used when we write:

$$dS_t = \mu_t S_t dt + a(S_t, t) dW_t, \quad t \geq 0.$$
Limiting Normalized Calendar Spreads

- Consider the initial price of a calendar spread of 2 calls with the same strike price $K$:
  \[ CS_0(K, T) = C_0(K, T + \triangle T) - C_0(K, T). \]

- The bigger is $\triangle T$, the bigger is the cost of a calendar spread, so suppose that the positions are normalized by dividing by the difference in maturities:
  \[ NCS_0(K, T) = \frac{C_0(K, T + \triangle T) - C_0(K, T)}{\triangle T}. \]

- Now let the difference in maturities approach zero:
  \[ LNCS = \lim_{\triangle T \downarrow 0} \frac{C_0(K, T + \triangle T) - C_0(K, T)}{\triangle T} \equiv \frac{\partial C_0(K, T)}{\partial T}. \]

- The result is the initial price of a limiting normalized calendar spread, or more simply a calendar spread.

- If one has a mechanism for generating implied volatility surfaces, one can observe $\frac{\partial C_0(K, T)}{\partial T}$ at time 0.
• Assume that the UK riskfree rate is zero for simplicity and take one pound as the numeraire.

• The first fundamental theorem of asset pricing says that under no arbitrage, there exists a risk-neutral measure $\mathbb{Q}$ such that the (pound denominated) price of each non-dividend paying asset is the expected value under $\mathbb{Q}$ of its payoff.

• Consider a claim that pays one pound at $T$ if the stock price at $T$ is below some constant $K$. The forward price of this claim is the risk-neutral probability that $S_T < K$.

• Let $q_0(K, T)$ denote the risk-neutral probability density function (PDF) observed at time $0$. If it exists, this function gives the rate at which the above forward price changes as we vary $K$:

$$q_0(T, K)dK \equiv Q\{S_T \in (K, K + dK) | \mathcal{F}_0\}.$$
Limiting Normalized Butterfly Spreads

• From Breeden and Litzenberger (1978), the risk-neutral PDF $q_0(T, K)$ can be obtained from the initial market price of a limiting normalized butterfly spread:

$$q(0, S_0; T, K) = \frac{\partial C_0(K, T)}{\partial K^2} = \lim_{\triangle K \downarrow 0} \frac{C_0(K + \triangle K, T) - 2C_0(K, T) + C_0(K - \triangle K, T)}{(\triangle K)^2}.$$ 

• As with the calendar spread, the initial price of the limiting normalized butterfly spread can be observed initially and is just referred to as a butterfly spread.

• In a Markovian setting, one can work more simply with the risk-neutral transition probability density function:

$$q(t, S; T, K) dK \equiv \mathbb{Q}\{S_T \in (K, K + dK) | S_t = S\}.$$
The Market’s Forecast of Local Volatility

- For simplicity, we assume no dividends in addition to zero interest rates.
- Under the assumptions made, the ratio of the calendar spread to the butterfly spread can be interpreted as the market’s perfect forecast of half the absolute variance rate, conditional on the future spot price being at $K$ at $T$.

$$\frac{\partial C_0(K,T)}{\partial T} \frac{\partial^2 C_0(K,T)}{\partial K^2} = \frac{1}{2} a^2(K, T).$$

- The next page starts to explains why.
- The assumptions imply that the RHS is independent of the date 0 at which the market makes its forecast.
- To obtain the market forecast of local relative vol instead, double the ratio, take the square root and divide by $K$. 
Why can Local Variance be Forecasted?

- Let $a^2(K, T)$ denote the forecast of local variance. Recall that Dupire assumed that this function does not vary through calendar time, so no time subscript on $a^2(K, T)$ is needed.

- By re-arranging: 
  $$\frac{\partial C_0(K, T)}{\partial T} = \frac{1}{2} a^2(K, T) \frac{\partial^2 C_0(K, T)}{\partial K^2},$$
  which is called Dupire’s forward PDE.

- Consider the payoff from holding the (limiting normalized) calendar spread on the LHS to the earlier maturity $T$.

- Paths which finish strictly below $K$ provide zero payoff. Similarly, paths which finish strictly above $K$ also provide zero payoff.

- Paths which finish at $K$ actually have a payoff which is infinite. If we condition on $S_T = K$, then the bigger is the market’s forecast at time 0 of the volatility to be experienced over $(T, T + dT)$, the larger is the conditional value of the calendar spread at $T$ and hence the larger is the unconditional value initially.
More Math

• Recall Dupire’s forward PDE: \( \frac{\partial C_0(K,T)}{\partial T} = \frac{1}{2} a^2(K,T) \frac{\partial^2 C_0(K,T)}{\partial K^2} \). To prove it, one can integrate the Kolmogorov forward p.d.e. in the forward spatial variable twice (see Dupire’s 1994 Risk paper).

• An alternative approach uses a generalization of Itô’s lemma to nondifferentiable convex functions such as \( f(S) = (S - K)^+ \).

• Yet, a third approach is to consider a binomial model and take limits. Over the time interval \((T, T + \Delta T)\) and conditioning on \( S_T = K \), we have:

\[
K \left\{ \begin{array}{c}
K + a(K,T) \sqrt{\Delta t} \\
K - a(K,T) \sqrt{\Delta t}
\end{array} \right.
\]

• To make \( S \) a martingale, the risk-neutral probabilities are both \( \frac{1}{2} \).
Binomial Derivation

\[ K + a(K, T) \sqrt{\Delta t} \]

\[ K - a(K, T) \sqrt{\Delta t} \]

Recall: \( K \)

- Since \( q_u = \frac{1}{2} \) and \( r = 0 \), the calendar spread at \( T \) given \( S_T = K \) has value:
  \[
  \frac{C_T(K, T + \Delta t) - C_T(K, T)}{\Delta t} = \frac{1}{2} \frac{a(K, T) \sqrt{\Delta t}}{\Delta t} = \frac{1}{2} \frac{a(K, T)}{\sqrt{\Delta t}}.
  \]

- If the stock price at \( T \) is at \( K - 2a(K, T) \sqrt{\Delta t} \) or below, then the calendar spread struck at \( K \) is worthless since both calls in it must expire OTM.

- If the stock price at \( T \) is at \( K + 2a(K, T) \sqrt{\Delta t} \) or above, then the calendar spread struck at \( K \) is also worthless since both calls in it must expire ITM.

- Thus, the calendar spread struck at \( K \) sells at \( T \) for \( \frac{1}{2} \frac{a(K, T)}{\sqrt{\Delta t}}1(S_T = K) \).
• Consider the payoff of a normalized butterfly spread maturing at $T$:

$$
\text{Stock Price at } T
\begin{align*}
&K - a(K, T)\sqrt{\Delta t} \\
&K + a(K, T)\sqrt{\Delta t}
\end{align*}
$$

Figure 1: Payoff at $T$ of a Butterfly Spread.
• Since the strike $K$ is attainable by the binomial process, the strikes $K \pm a(K, T) \triangle t$ are not attainable by the process.
• Note that the area under the triangular payoff is one.

\[
\text{Area} = \frac{1}{2} \times \text{Base} \times \text{Height}
\]
\[
= \frac{1}{2} \times 2a(K, T)\sqrt{\triangle t} \times \frac{1}{a(K, T)\sqrt{\triangle t}}
\]
\[
= 1.
\]

• The height of the triangular payoff is $\frac{1}{a(K, T)\sqrt{\triangle t}}$. If an investor is long one butterfly spread and the stock price hits $K$ at $T$, then the investor gets $\frac{1}{a(K, T)\sqrt{\triangle t}}$ pounds. If $S_T \neq K$, the investor gets nothing.
Binomial Derivation (Con’d)

- Suppose an investor holds \( \frac{a^2(K,T)}{2} \) butterfly spreads. If the stock price hits \( K \) at \( T \), then the investor gets \( \frac{a^2(K,T)}{2} \times \frac{1}{a(K,T)\sqrt{\Delta t}} = \frac{1}{2} \frac{a(K,T)}{\sqrt{\Delta t}} \) pounds. If \( ST \neq K \), the investor gets nothing.

- Recall that the calendar spread liquidated at \( T \) pays \( \frac{1}{2} \frac{a(K,T)}{\sqrt{\Delta t}} 1(ST = K) \) pounds.

- Hence this payoff can be duplicated by holding \( \frac{a^2(K,T)}{2} \) butterfly spreads from time 0 to time \( T \).

- It follows from no arbitrage that
  \[
  \frac{C_0(K,T+\Delta t) - C_0(K,T)}{\Delta t} = \frac{1}{2} a^2(K,T) \frac{C_0(K + \Delta K, T) - 2C_0(K, T) + C_0(K - \Delta K, T)}{(\Delta K)^2},
  \]
  where \( \Delta K = a(K,T)\sqrt{\Delta t} \).

- In the diffusion limit, we have Dupire’s forward PDE:
  \[
  \frac{\partial C_0(K, T)}{\partial T} = \frac{1}{2} a^2(K, T) \frac{\partial^2 C_0(K, T)}{\partial K^2}.
  \]
If we add a deterministic riskfree rate \( r(t) \) and a deterministic dividend yield \( q(t) \), Dupire’s PDE generalizes to the following forward PDE for European calls:

\[
\frac{\partial C_0}{\partial T} = \frac{1}{2} a^2(K,T) \frac{\partial^2 C_0}{\partial K^2} - [r(T) - q(T)] K \frac{\partial C_0}{\partial K} - q(T) C_0.
\]

One interpretation of the extra terms is that they represent the stock carrying cost which must be paid when the stock price is above the strike at the earlier maturity. By re-arrangement:

\[
\frac{\partial C_0}{\partial T} + [r(T) - q(T)] K \frac{\partial C_0}{\partial K} + q(T) C_0 = \frac{K^2}{2} a^2(K,T) \frac{\partial^2 C_0}{\partial K^2}.
\]

The LHS is the cost of a ratioed calendar spread. Its value is nonnegative for any stochastic process for the underlying.

The exact same PDE holds for European puts or any payoff with a single kink.
Recall Dupire’s forward PDE for calls under a deterministic riskfree rate $r(t)$ and a deterministic dividend yield $q(t)$:

$$\frac{\partial C_0}{\partial T} + [r(T) - q(T)]K\frac{\partial C_0}{\partial K} + q(T)C_0 = \frac{a^2(K,T)}{2} \frac{\partial^2 C_0}{\partial K^2}.$$ 

To express local volatility in terms of option prices, solve the forward PDE for $a(K,T)$:

$$a(K,T) \equiv \sqrt{\frac{2 \left\{ \frac{\partial C_0(K,T)}{\partial T} + [r(T) - q(T)]K\frac{\partial C_0(K,T)}{\partial K} + q(T)C_0(K,T) \right\}}{\frac{\partial^2 C_0(K,T)}{\partial K^2}}}.$$ 

To express local volatility in terms of implied volatility rather than option prices, substitute in the Black Scholes formula and carry out the differentiations.
Valuation of Derivatives

- Since the stock price process is continuous and driven by a single Brownian motion, one can theoretically replicate the payoff on any claim by dynamic trading in stock and bond.

- Since the stock price process is Markov in itself and time, the value function $V(S,t)$ for many claims (e.g. standard American options or barrier options) solves the following backward PDE:

$$
\frac{\partial V(S,t)}{\partial t} + \frac{a^2(S,t) \frac{\partial^2 V(S,t)}{\partial S^2}}{2} + [r(t) - q(t)]S \frac{\partial V(S,t)}{\partial S} = r(t)V(S,t).
$$

- By appending the appropriate boundary conditions, one can numerically solve the PDE using say finite differences.
• Recall that $V(S, t)$ is the value function obtained by numerically solving a backward PDE.

• To obtain delta, the standard approach is to use the following centered finite difference approximation:

$$
\frac{\partial}{\partial S}V(S_0, t_0) = \frac{V(S_0 + \Delta S, t_0) - V(S_0 - \Delta S, t_0)}{2\Delta S} + O(\Delta S^2).
$$

• One can alternatively differentiate the backward PDE w.r.t. $S$ to get a PDE for delta.

• To obtain the deltas of European options, one can just differentiate the forward PDE. The form of the PDE does not change and one obtains deltas for all strikes and maturities with a single pass.

• The same idea holds for thetas and gammas of European options (which theoretically one does not need to know to carry out a delta hedge).
Both the parametric and non-parametric forms of price-dependent instantaneous volatility models make strong predictions which can be tested.

For example when interpreted literally, both approaches require only knowledge of the stock price and the time in order to predict the entire implied volatility surface at each time after the initial calibration.

In order to test the validity of the model, this prediction can be compared to the implied volatilities obtained from market prices of options.

The general finding from index options is that local vol models predict future short term skews which have weaker dependence on strike than is found in the data.
In practice, models are re-calibrated each day, and as a result, it is reasonable to ask how well the re-calibrated versions of these models perform.

Models with continuous price dynamics make very specific predictions about the average change in the implied volatility surface for a given change in the stock price and time. When the stock price process is also Markovian in just itself and time, the prediction is not just on average but is made path by path.

To eliminate variance just by dynamic delta hedging, we need no other state variables, no price jumps, and a correct prediction of how the implied volatility surface moves with (small) changes in stock price and time.

For example, suppose that implied volatility slopes down in $K$ at each $T$. Suppose $S = K = 100$. If $S$ moves up, then the local vol model predicts that the implied for strike 100 moves down. If this implied actually rises, then using the local vol delta injects more variability into the P&L than a model that correctly predicts surface changes.
Approach 3: Lévy Processes

- In this segment, I will quickly survey the link between European option pricing and characteristic functions.

- I will then introduce Lévy processes and provide a selective survey of some option valuation models which use non-Gaussian Lévy processes to model returns.

- I apologize in advance to authors whose work I miss; to simply list all the work on this subject would fill up the segment.

- Much of what I review can be downloaded from www.math.nyu.edu/research/carrp/papers
Let $S$ be a positive stock price process. Let $s \equiv \ln S_T$ denote the log of the terminal stock price and $k \equiv \ln k$ denote the log of the strike.

The standard approach to European option valuation expresses the option value $c(k; T)$ as an expectation of its discounted payoff:

$$c(k; T) \equiv e^{-rT} \int_{k}^{\infty} (e^s - e^k) q(s; T) \, ds.$$  

This approach requires that the risk-neutral density $q(s; T)$ be known and fast valuation requires that it be simple.
European Option Pricing & Fourier Methods

- Fourier methods describe an array of techniques that can be used to obtain option prices.

- For fixed maturity $T$, consider the call value $c$ as a function of the log of its strike $k$. If its Fourier transform exists, then this function is given by:
  \[ \psi(v; T) \equiv \int_{-\infty}^{\infty} e^{ivk} c(k; T) dk. \]

- By the Fourier inversion theorem, the call value is the inverse of its Fourier transform:
  \[ c(k; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi(v; T) dv. \]

- This approach requires that the Fourier transform $\psi(v; T)$ be known and fast valuation requires that it be simple.
Recall that $q(s; T)$ denotes the risk-neutral (RN) probability density function (PDF) of log spot $s_T \equiv \ln S_T$.

When $q$ is considered as a function of $s$ for fixed maturity $T$, then its Fourier transform (FT) is called the characteristic function (CF) of $s_T$:

$$\phi(v; T) \equiv E e^{ivs_T} = \int_{-\infty}^{\infty} e^{ivs} q(s; T) ds.$$ 

By the Fourier inversion theorem, the PDF is obtained by inverting the characteristic function of $s_T$:

$$q(s; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivs} \phi(v; T) dv.$$
Obtaining Option Values at Many Strikes

• Recall that $\psi(v; T) \equiv \int_{-\infty}^{\infty} e^{ivk} c(k; T) dk$ denotes the Fourier Transform (FT) of the call price $c(k; T)$ and that $\phi(v; T) \equiv E e^{ivs_T}$ denotes the characteristic function (CF) of $s_T \equiv \ln S_T$.

• Carr Madan JCF99 shows how to analytically relate the FT of the call price $\psi(v; T)$ to the CF $\phi(v; T)$.

• We then numerically invert this FT using the Fast Fourier Transform (FFT). Thus, the option valuation problem is reduced to determining the CF of the terminal log price.

• Our approach outputs a vector of option prices at many strikes, so our approach is particularly fast when this is the output needed.
When the log price is a Lévy process, then its characteristic function is explicitly given by the Lévy-Khintchine Theorem.

Thus one can numerically value all options written on exponential Lévy processes, even if the transition density is unknown.

If the transition density is known, our numerical results show that it may still be faster to work in Fourier space, as some stochastic processes generate terminal random variables whose characteristic function is simpler than its PDF (eg. Variance Gamma).
Fourier Transform of Dampened Call Value

• Let $c(k; T)$ be the value of a $T$ maturity call with log strike $k$. Let $q(s; T)$ denote the RN PDF of the terminal log price $s_T$. The CF of this PDF is:

$$
\phi(\omega; T) \equiv \int_{-\infty}^{\infty} e^{i\omega s} q(s; T) ds.
$$

• The initial call value $c(k; T)$ is related to the risk-neutral density $q(s; T)$ by:

$$
c(k; T) \equiv e^{-rT} \int_{k}^{\infty} (e^s - e^k) q(s; T) ds.
$$

• The function $c$ is not square integrable in $k$, so we Fourier transform the dampened call price, $e^{\alpha k} c(k; T)$, $\alpha > 0$:

$$
\psi(\omega; T) \equiv \int_{-\infty}^{\infty} e^{i\omega k} e^{\alpha k} c(k; T) dk = \ldots = \frac{e^{-rT} \phi(\omega - (\alpha + 1)i; T)}{\alpha^2 + \alpha - \omega^2 + i(2\alpha + 1)\omega}.
$$

• Calls are valued by inverting both the FT and the dampening factor:

$$
c(k; T) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega k} \psi(\omega; T) d\omega = \frac{\exp(-\alpha k)}{\pi} \int_{0}^{\infty} e^{-i\omega k} \psi(\omega; T) d\omega.
$$
Terminology 101: Stable vs Lévy Processes

- Probabilists and physicists use the same words to describe different things. I will use probabilistic language in what follows.

- In words, Lévy processes are the class of continuous time stochastic processes with stationary independent increments.

- In words, stable processes are a subset of Lévy processes which loosely speaking have power law tails. The formal definition expresses the Lévy density as a particular function of the jump size.

- Physicists often refer to stable processes as Lévy flights or Lévy processes.
A random variable has a symmetric stable distribution if all odd moments vanish and tail behavior is governed by the tail parameter $\alpha \in (0, 2]$.

If random variable $X$ is distributed symmetric stable($\alpha$), then $EX^p$ exists for $p \in [0, \alpha]$ but not for $p > \alpha$. A standard normal arises when $\alpha = 2$.

As usual, one can shift and scale a stable random variable and a skewness parameter $\beta \in [-1, 1]$ can also be introduced.

For $\alpha \in (0, 2)$, the variance of $X$ is always infinite, but the quadratic variation of a stable process is finite path by path.
Lévy Processes and Lévy Khintchine Theorem

- By the Lévy Khintchine theorem, the characteristic function (FT of PDF) of the random variable $s_T$ when $s$ is a Lévy process (i.e. right continuous left limits processes with stationary independent increments) is given by:

$$Ee^{i\omega s_T} = \exp \left\{ T \left[ i\hat{\mu} \omega - \frac{\sigma^2 \omega^2}{2} + I_1(\omega) + I_2(\omega) \right] \right\},$$

where:

$$I_1(\omega) \equiv \int_{|j| \geq 1} (e^{i\omega j} - 1) \ell(dj)$$

$$I_2(\omega) \equiv \int_{0<|j|<1} (e^{i\omega j} - 1 - i\omega j) \ell(dj)$$

and:

$$\hat{\mu} = \mu - \int_{|j| \geq 1} \frac{j}{1 + j^2} \ell(dj) - \int_{0<|j|<1} \left( \frac{j}{1 + j^2} - j \right) \ell(dj).$$

- The Lévy process is specified by the drift $\mu$, the diffusion coefficient $\sigma$, and the Lévy measure $\ell(dj)$. 

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Examining 3 Non Gaussian Lévy Models

- The first non-Gaussian Lévy model is due to Cox and Ross (1976) in which the log forward price occasionally jumps up by a constant and drifts down deterministically to compensate. They showed that an option could be replicated by continuous trading in the underlying stock and bond, using the same intuition as for the binomial model. Ditschel and Protter (1999) extend this idea to Azema martingales.

- Merton (1976) generalized the simple Cox-Ross model by letting the jump in the log be normally distributed, and adding a Brownian motion. Kou (1999) lets the jump in the log be double exponential (Laplace) rather than normal.

- Some recent work has focussed on modelling log stock prices as pure jump processes with infinite activity. The empirical work shows that the addition of a diffusion to these processes has no additional explanatory power. Yet these models can converge to Black Scholes as a limit.
Hedging in Jump Models

- When the jump size distribution is continuous as in Merton, the only way to replicate an arbitrary nonlinear payoff in a Markovian setting is by dynamic trading in a single continuum of options on the same stock.

- Such jump models usually assume a finite number of options which are taken to be primary assets. Thus, there are many risk-neutral price processes which are consistent with the observed initial asset prices, an assumed statistical process for the stock price, and no arbitrage.

- The selection of a particular risk-neutral process for stock returns is a way to indirectly select a particular risk-neutral process and a corresponding initial value for each member of the family of options written on the stock.

- The validity of the choice can be measured by the extent to which the option prices predicted by the model match the observed market option prices across all strikes, maturities, and dates.
The Variance Gamma Model

- The Variance Gamma option pricing model (henceforth VG) is due to Dilip Madan and various co-authors (including myself).

- In VG, the driving SBM in the Black Scholes model is replaced by an arithmetic Brownian motion (ABM) subordinated to an unbiased gamma process.

- The volatility parameter of the Black Scholes model still controls the standard deviation of returns. The drift of the ABM controls return skewness. The variance rate of the gamma process controls return kurtosis.

- There exists closed form formulas for the characteristic function of returns, the terminal stock price PDF, and European option values.

- For American and barrier option values, forward and backward partial integro differential equations (PIDE’s) can often be solved numerically, or else the risk-neutral price process can be simulated.
More on the VG Model

- In VG, the driver of the log price is a Lévy process obtained by evaluating arithmetic Brownian motion with drift $\theta$ and volatility $\sigma$ at a random time given by a gamma process having a mean rate per unit time of 1 and a variance rate of $\nu$.

- The resulting process $X_t(\sigma, \theta, \nu)$ is a pure jump process with two additional parameters $\theta$ and $\nu$ relative to the Black Scholes model, providing control over skewness and kurtosis respectively.

- The risk-neutral process for the stock price is:
  \[ S_t = S_0 e^{r t + X_t(\sigma, \theta, \nu) + \eta t}, t \in [0, T], \]
  where by setting $\eta = \frac{\ln(1 - \theta \nu - \sigma^2 \nu / 2)}{\nu}$, the mean rate of return on the stock equals the riskfree rate $r$.

- The characteristic function for the log of $S_T$ is simply:
  \[ \phi_T(\omega) = \frac{S_0 e^{(r + \eta) T}}{(1 - i \theta \nu \omega + \sigma^2 \omega^2 \nu / 2)^{T/\nu}}. \]
Even More on the VG Model

• Recall that the characteristic function for the log of $S_T$ is:

$$
\phi_T(\omega) = \frac{S_0 e^{(r+\eta)T}}{(1 - i\theta \nu \omega + \sigma^2 \omega^2 \nu / 2)^{T/\nu}}.
$$

• A closed form option pricing formula can be obtained by analytically inverting this characteristic function to get the density function and then integrating this density function against the option payoff.

• Alternatively, the Fourier transform of the distribution functions can be inverted analytically.

• As with any model where the CF of the log price is known in closed form, FFT can be employed to numerically determine European option prices as a function of strike.
The price path of the standard geometric Brownian motion reflects no jumps and displays unbounded variation. In contrast, the Variance Gamma (VG) price path reflects infinite jump activity and has bounded variation. Both processes have finite quadratic variation, i.e. are semi-martingales.

The CGMY model (due to Carr, Géman, Madan, and Yor (2002) generalizes the 3 parameter VG driver by adding a fourth parameter, which controls this fine structure of price paths.

By tuning the fourth parameter, the path can reflect finite or infinite activity, bounded or unbounded variation, and finite or infinite quadratic variation.
The CGMYe Model

• The CGMYe model extends CGMY by adding a diffusion component $\sigma W$ to the return process.

• There exists a closed form formula for the characteristic function of the CGMYe process. Thus, European option values can be obtained easily by the FFT approach of Carr and Madan.

• The implied $\sigma$ parameter estimates to zero in our empirical tests! Option prices need not reflect diffusion.

• The other parameters implied by our empirical work suggest that the stock price path reflects infinite jump activity, bounded variation, and finite quadratic variation.
The Term Structure of the Smile

- In many options markets, the implied volatility smile/smirk flattens out as maturity increases.

- The conventional wisdom is that the Central Limit Theorem has kicked in:

  to quote a statement of Poincaré, who said (partly in jest no doubt) that there must be something mysterious about the normal law since mathematicians think it is a law of nature whereas physicists are convinced that it is a mathematical theorem.

Mark Kac

*Statistical Independence in Probability Analysis and Number Theory*  
Chapter 3, The Normal Law (p. 52)
The S&P500 Volatility Surface

- S&P500 index option implieds do NOT “wipe that smirk off your face”.

- Here is a nonparametrically smoothed surface from April 4th, 1999 to May 31st, 2000:
The S&P500 Volatility Surface (Con’d)

• It is important that the horizontal axis uses “moneyness” defined as:

\[
\text{Moneyness } d = \frac{\ln K/F}{\sigma \sqrt{\tau}}
\]

i.e. \( \approx \) # standard deviations that log strike is away from log forward.

• In all option pricing models for which the stock price is driven by a Lévy process with finite moments (eg VG and CGMY), the volatility smile flattens with increasing maturity.

• The introduction of a stationary stochastic volatility process slows this flattening, but does not stop it.
To capture the behavior of S&P500 options, Carr and Wu (2002) price options using the “Finite Moment Log Stable Process” (FMLS)
– with unbounded variance, skewness, kurtosis of returns
– yet finite expectation of price, price^2, price^3 etc,

This balancing act is achieved by setting the skewness parameter $\beta = -1$. The smile’s level is controlled by $\sigma > 0$ and its slope is controlled by $\alpha \in (0, 2]$. Options are again priced via FFT.

Calibration of our FMLS model to the option price data suggests that:
– it outperforms other stationary jump (-diffusion) models
– by capturing the stability of the volatility smirk across maturity.
Maturity Pattern: 1m (solid), 6m (dashed), 12m (dash-dotted)
References


Approach #4: Local Lévy

- At present, the theory of arbitrage pricing has been worked out when asset prices are semi-martingales.

- Intuitively, a semi-martingale is a generalization of a Lévy process where the 3 characteristics are allowed to be random processes called the triple $(B, C, \nu)$. Hence, a semi-martingale can always be decomposed into the sum of the following 3 types of processes:
  1. a process of bounded variation $B_t$, which in the Lévy case is just $b \times t$
  2. a continuous local martingale $M^c$ with a random (increasing) quadratic variation process $C_t$. In the Lévy case, $C_t = \sigma^2 t$.
  3. a pure jump martingale $M^j$ with a random but predictable compensator $\nu_t$. In the Lévy case with finite variation sample paths, $\nu_t = \int_{-\infty}^{\infty} xK(dx) \times t$, where $K$ is the Lévy measure.
Stochastic Differential Equations

• If we assume for tractability that the components of the triple describing the log price are all absolutely continuous w.r.t. time, then using the language of SDE’s, increments in the stock price $S$ process decompose as:

$$dS_t = b_t S_t dt + \sigma_t S_t dW_t + \int_{-\infty}^{\infty} S_t - (e^x - 1) \left[ \mu_t(dx) - K_t(dx) \right] dt,$$

where the jump process has been assumed to display finite variation for ease of exposition.

• If we are working under the usual risk-neutral measure $Q$, then the (relative) risk-neutral drift $b_t$ is just the difference between the spot interest rate $r_t$ and the dividend yield $q_t$. For simplicity, we will treat these as constant at $r$ and $q$ respectively.
Markovian Stock Price Dynamics

- Under the assumptions of the last page: increments in the stock price $S$ process decompose as:

$$dS_t = (r - q)S_t - dt + \sigma_t S_t - dW_t + \int_{-\infty}^{\infty} S_t - (e^x - 1) \left[ \mu_t(dx) - K_t(dx) \right] dt,$$

- Hence, modelling under the risk-neutral measure $\mathbb{Q}$ amounts to specifying the martingale component of the stock part and if we further assume that $S$ is Markovian in itself and time, then increments in the stock price $S$ process decompose as:

$$dS_t = (r - q)S_t - dt + a(S_t, t) dW_t + \int_{-\infty}^{\infty} S_t - (e^x - 1) \left[ \mu_t(dx) - K(S_t, t, dx) \right] dt.$$

- Hence, the risk-neutral Markovian stock price process $S$ is specified once we pick a “volatility function” $a(S, t)$ and a jump compensation measure $K(S, t, dx)$. 

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Forward-Start Claims

- We define forward-start claims as the class of contingent claims whose terminal payoff is a nonlinear function of future returns $\ln(S_{T_j}/S_{T_i})$ for $T_j > T_i > 0$.

- The classical example of a forward-start claim is an at-the-money forward-start call whose payoff at $T_2$ is $(S_{T_2} - S_{T_1})^+$. 

- The prices of forward-start claims are sensitive to the risk-neutral distribution of future returns $\ln(S_{T_j}/S_{T_i})$.

- A model which is consistent with standard option prices need not produce risk-neutral distribution of future returns $\ln(S_{T_j}/S_{T_i})$ which are consistent with market prices of forward-start claims.
• Suppose we wish to value an at-the-money forward-start call whose pay-off at $T_2$ is $(S_{T_2} - S_{T_1})^+$.

• Suppose that a single factor Markovian model has been calibrated to the time 0 market prices of standard options maturing at $T_2$.

• One can use the model to determine the risk-neutral distribution of the future return $\ln(S_{T_2}/S_{T_1})$ conditional on some future stock price $S_{T_1}$. If for every choice of $S_{T_1}$, the resulting return distribution lies outside the range of distributions ever observed over a time horizon of length $T_2 - T_1$, one has to wonder if the right model was selected.

• Local Lévy models were suggested by Carr, Géman, Madan, and Yor (2004) as a way to provide direct control on forward return distributions, while retaining consistency with initial options prices.
In defining a local Lévy option pricing model, Carr, Géman, Madan and Yor (2004) eliminate the continuous martingale component of the risk-neutral Markovian stock price process $S$ by setting $a(S,t) = 0$. In other words, they assume that there is no continuous martingale component and hence all uncertainty is generated solely by jumps.

CGMY further propose specifying the jump compensation measure $K(S,t,dx) = b(S,t) \times k(x)dx$, where $b(S,t) : \mathbb{R}^+ \times [0,T] \mapsto \mathbb{R}^+$ is called the local speed function and $k(x)$ meets all of the mathematical requirements of a Lévy density.

They suggest that $k(x)$ be determined ex ante as say the Lévy density of a Variance Gamma process, while the local speed function $b$ is determined by requiring that the model value of all European options be consistent with an assumed given complete term and strike structure of European option market prices.
To speed up the determination of the local speed function $b(S,t)$, CGMY (2004) derive a forward PIDE governing European option prices.

In general, the local speed function must be determined by solving a convolution integral equation. Hence in general the local speed function is found by applying an integral operator to the option data.

However, special choices for the local Lévy density $k(x)$ (eg. Kou model) permit inversion by applying a differential operator. As a result, the local speed function is expressed by a local operation on option prices just as in the original Dupire PDE.

Although local Lévy dynamics can be used to price any univariate contingent claim, the model is designed to correctly price forward-start claims, assuming that (a rather expert) trader can directly specify local Lévy density $k(x)$ in order to exert direct control on forward skews.
In the second part of this morning’s presentation, we review ≤ 3 approaches for pricing options which are Markovian in two stochastic state variables:

1. Traditional SV models: Continuous stock price models, where the instantaneous variance evolves autonomously and where correlation between returns and increments in instantaneous variance is constant.
2. Time-Changed Lévy Processes: a generalization of the above, where the continuity requirement is relaxed.
3. Time-Changed Markov Processes: an independent time change of a tractable time-homogeneous Markov process such as CEV.

This list is not exhaustive! For example, one could consider time-changing a local Lévy process.
Approach #1: Instantaneous Volatility is an Autonomous Diffusion

- Suppose once again that the risk-neutral process is:
  \[ dS_t = rS_t dt + a(S_t, t) dW_t, \quad t \in [0, T], \]

- By Itô’s lemma, the stock volatility \( a_t \equiv a(S_t, t) \) follows a diffusion process:
  \[ da_t = \mu(S_t, t) dt + \omega(S_t, t) dW_t, \quad t \in [0, T], \]
  for some functions \( \mu(\cdot) \) and \( \omega(\cdot) \).

- If we suppose further that the map between \( a \) and \( S_t \) is invertible, then the stock volatility \( a_t \) follows an autonomous diffusion process:
  \[ da_t = m(a_t, t) dt + v(a_t, t) dW_t, \quad t \in [0, T], \]
Instantaneous Volatility is an Autonomous Diffusion Process

- Recall that when the instantaneous vol \( a \) is a function of stock price, we have:
  \[
d a_t = m(a_t, t)dt + v(a_t, t)dW_t, \quad t \in [0, T],
\]
  where \( W \) also drives the stock.

- In the late 1980’s and early 90’s, several researchers extended this approach to:
  \[
da_t = m(a_t, t)dt + v(a_t, t)dB_t, \quad t \in [0, T],
\]
  where \( B \) is a second Brownian motion, which may or may not be correlated with \( W \).

- For example, Stein and Stein (1991) consider the case where the lognormal volatility \( \sigma_t \equiv \frac{a_t}{s_t} \) is an OU process:
  \[
da \sigma_t = \delta(\theta - \sigma_t)dt + kdB_t, \quad t \in [0, T].
\]
• Let $V_t \equiv a_t^2$ denote the absolute variance process.
• Closed form formulas for option prices were given by:
  – Hull and White (1987):
    $$dV_t = \alpha V_t dt + \xi V_t dB_t, \quad t \in [0, T].$$
  – Heston (1993)
    $$dV_t = (\omega - \theta V_t) dt + \xi \sqrt{V_t} dB_t, \quad t \in [0, T].$$
• A review article by Ball and Roma (1994) surveys these approaches.
• Ritchken and Trevor (1998) showed that the Hull-White model emerges as a continuous time limit of a GARCH 1-1 process.
A book called “Option Valuation under Stochastic Volatility” by Alan Lewis extends the Hull White model to correlated Brownian motions, derives the Heston model, and introduces a “3/2” model:

\[ dV_t = (\omega V - \tilde{\theta} V_t^2) dt + \xi V_t^{3/2} dB_t, \quad t \in [0, T], \]

where \( \tilde{\theta} \) is a certain function of the parameters.

- The publisher of this book is “Finance Press” at “www.financepress.com”.

- The book presents the argument for treating the statistical process for S&amp;P500 volatility as a fast mean reverting process.
• When volatility follows an autonomous diffusion, the “market price of volatility risk” is usually specified to take some tractable form.

• The emergence of this concept has lead many (respectable) researchers to mistakenly conclude that markets are not complete when volatility follows an autonomous diffusion. They then assume some equilibrium model to justify their analysis.

• In fact, markets are complete so long as one can trade continuously in another option on the same stock. Thus, standard arbitrage pricing arguments can be used to develop “preference-free” option pricing models.

• The key to avoiding preference restrictions is to relate option prices to either other option prices or to known functions of option prices such as “forward local volatility” or Black Scholes implied volatility.

• To illustrate these points, I now show how the market price of volatility risk drops out when an option price is related to another option price.
Eliminating the Market Price of Vol Risk

• Using standard arguments, one can derive the following PDE governing the function \( C^{(1)}(t, S, Y) \) relating the price of an option to time \( t \), the price of the underlying stock \( S \), and the price of a state variable governing volatility \( Y \):

\[
\frac{\partial C^{(1)}}{\partial t} + r \left[ S \frac{\partial C^{(1)}}{\partial S} - C^{(1)} \right] + \left[ \alpha(m - Y) - \lambda \beta \right] \frac{\partial C^{(1)}}{\partial Y} + \frac{f^2(Y)}{2} \frac{\partial^2 C^{(1)}}{\partial S^2} + \rho f(Y) \beta \frac{\partial^2 C^{(1)}}{\partial S \partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(1)}}{\partial Y^2} = 0,
\]

where \( \lambda \) is the market price of volatility risk.

• The above PDE also holds for the price of a second option:

\[
\frac{\partial C^{(2)}}{\partial t} + r \left[ S \frac{\partial C^{(2)}}{\partial S} - C^{(2)} \right] + \left[ \alpha(m - Y) - \lambda \beta \right] \frac{\partial C^{(2)}}{\partial Y} + \frac{f^2(Y)}{2} \frac{\partial^2 C^{(2)}}{\partial S^2} + \rho f(Y) \beta \frac{\partial^2 C^{(2)}}{\partial S \partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(2)}}{\partial Y^2} = 0.
\]
Eliminating the Market Price of Vol Risk

• Let $\gamma(t, S, C^{(2)})$ be a function relating the price of the 1st option to time $t$, the stock price $S$, and the price of the 2nd option:

$$\gamma(t, S, C^{(2)}) \equiv C^{(1)}(t, S, Y),$$

where $C^{(2)}$ solves the above PDE.

• The appendix proves:

$$\begin{align*}
\gamma_1 + rS\gamma_2 + rC^{(2)}\gamma_3 \\
+ \frac{1}{2} \frac{\text{Var}(dS)}{dt}\gamma_{22} + \frac{\text{Cov}(dC^{(2)}, dS)}{dt}\gamma_{23} + \frac{1}{2} \frac{\text{Var}(dC^{(2)})}{dt}\gamma_{33} = r\gamma.
\end{align*}$$

• If one can exogenously model the volatility structure of $C^{(2)}$, then one does not need to specify $\lambda$. There are many ways to do this.
References


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[29] Hilliard, J. and A. Schwartz, 1994, Binomial Option Pricing Under Stochastic Volatility and Correlated State Variables, University of Geor-


[38] Lazrak, A., 1997, General equilibrium foundation of the stochastic volatility model: A theoretical investigation and an example, GREMAQ, Université des Sciences Sociales working paper.


[40] Liesen D., 1997, Stock evolution under stochastic volatility: A discrete time approach, CREST working paper.


399-412.


[61] Walsh D. and G. Tsou, 1997, Forecasting index volatility: Sampling interval and non-trading effects, University of Western Australia working paper.


Approach #2: Time-Changed Lévy Processes

- Apply stochastic time change to Lévy processes:
  - Lévy processes can generate non-normal return innovations.
  - Stochastic time changes generate stochastic volatility.
  - Correlation between the two captures the “leverage effect”.

⇒ Our framework encompasses almost all extant option pricing models and points to new directions for designing new models.

- What we do:
  - Derive the generalized characteristic function (CF) of the time-changed Lévy process.
  - Propose FFT algorithms to price European-style options via this generalized CF.
  - Specification analysis (model design, examples).
Related Literature

- Affine jump-diffusion stochastic volatility models of Duffie, Pan, Singleton (2000):
  - Finite-activity compound Poisson jumps: Jumps are regarded as rare events.
    Evidence: Asset prices display many small jumps on a finite time scale: ⇒ Infinite-activity jumps may perform better.
  - Affine volatility dynamics:
    A linear-quadratic structure is more flexible for incorporating correlations.

- Time-changed Lévy processes:
  - The Lévy process can accommodate both low and high frequency jumps.
  - Stochastic time change can accommodate both affine and quadratic volatility dynamics.
  - Stochastic volatility can be driven by stochastic diffusion variance or stochastic jump arrival rate, or both.
– Flexible correlations between return and volatility are possible.
Lévy Processes and the Lévy-Khintchine Formula

• $X$: a $d$-dimensional Lévy process: rcll (right continuous with left limits), stationary independent increments, stochastic continuity.
• Defined on a probability space $(\Omega, F, P)$ endowed with a standard complete filtration $F = \{F_t | t \geq 0\}$.
• The Lévy-Khintchine formula for the characteristic function of $X_t$:
  \[
  \phi_{X_t}(\theta) \equiv E \left[ e^{i\theta^\top X_t} \right] = e^{-t\Psi_x(\theta)}, \quad t \geq 0, \theta \in \mathbb{R}^d
  \]
  where the characteristic exponent $\Psi_x(\theta)$, $\theta \in \mathbb{R}^d$, is given:
  \[
  \Psi_x(\theta) \equiv -i\mu^\top \theta + \frac{1}{2} \theta^\top \Sigma \theta - \int_{\mathbb{R}^d - \{0\}} \left( e^{i\theta^\top x} - 1 - i\theta^\top x 1_{|x| < 1} \right) \Pi(dx).
  \]
• Lévy characteristics: $(\mu, \Sigma, \Pi)$, with $\mu$ a $d$-vector, $\Sigma$ a semi-definite symmetric $d \times d$ matrix, and $\Pi : \mathbb{R}^d - \{0\} \mapsto \mathbb{R}^+$ a measure with some integrability properties.
• The generalized Fourier transform (or CF): $\theta \in \mathcal{D} \subseteq \mathbb{C}^d$. 

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Stochastic Time Change

• Let $t \to T_t, t \geq 0$ be an increasing rcll stochastic process such that for each fixed $t$, the random variable $T_t$ is a stopping time with respect to $F_t$.

• $T_t$ is finite $P$-a.s. for all $t \geq 0$ and $T_t \to \infty$ as $t \to \infty$.

• The family of stopping times $\{T_t; t \geq 0\}$ defines a stochastic time change.

• Define $Y$ by evaluating $X$ at $T$, i.e.

$$Y_t \equiv X_{T_t}, \quad t \geq 0.$$ 

• We use the time changed Lévy process, $Y$, as the source of all uncertainty in the economy.
Lévy Subordinators

- The random time $T_t$ can be modeled as a nondecreasing semimartingale:

$$T_t = \alpha_t + \int_0^t \int_0^\infty z\mu(dz, ds)$$

- Example: Lévy subordinators as random time changes:

$$T_t = \int_0^t \int_0^\infty z\Pi(dz)ds$$

  - Bertoin (1999): A Lévy process time changed by a Lévy subordinator yields a new Lévy process.
  - We can always suppress the subordinator by directly specifying the appropriate Lévy characteristics for $X$. 
Business Activity Rates

- We focus on locally predictable time changes:
  \[ T_t = \alpha_t = \int_0^t \nu(s-)ds. \]

- We call \( \nu(t) \) the **instantaneous (business) activity rate**.

- Economic Interpretations:
  - \( t \) — calendar time; \( T \) — business time.
  - \( \nu(t) \) captures the intensity of the business activity at calendar time \( t \).

- If \( X \) is SBM, then \( \nu(t) \) is the instantaneous variance rate of \( Y_t \equiv X_{T_t} \).

- If \( X \) is a pure jump Lévy process, then \( \nu(t)\Pi(dy) \) is the arrival rate of a jump of size \( y \) in \( Y \).

- Although \( T_t \) is assumed to be continuous, \( \nu(t) \) can jump.
Encompassing Extant Models

- Heston (1993): $X_t = W_t$, $\nu(t)$ follows a mean-reverting square-root process.

- Hull and White (1987): $X_t = W_t$, $\nu(t)$ follows an independent log-normal process.

- Affine jump-diffusion of Duffie, Pan, Singleton (2000): $X_t$ is diffusion plus compound Poisson jumps; $\nu(t)$ follows affine dynamics.
Independent Time Change

• \( \nu(t) \) evolves independently of \( X_t \), e.g. Hull and White (1987), no leverage effect.

• The characteristic function of the time changed Lévy process \( Y_t = X_{T_t} \) is

\[
\phi_y(\theta) \equiv E e^{i\theta^\top X_{T_t}} = E \left[ E \left[ e^{i\theta^\top X_u} \mid \mathcal{F}_t^\nu \right] \right] = E \left[ E \left[ e^{i\theta^\top X_u} \mid T_t = u \right] \right]
\]

\[
= E e^{-T_t \Psi_x(\theta)} = \mathcal{L}_T(\Psi_x(\theta))
\]

which is the Laplace transform of the stochastic time \( T_t = \int_0^t \nu(s-) \, ds \), evaluated at the characteristic exponent of \( X \).

• To obtain the Laplace transform of \( T \) in closed form, consider its specification in terms of the activity rate:

\[
\mathcal{L}_T(\lambda) \equiv E \left[ \exp \left( -\lambda \int_0^t \nu(s_-) \, ds \right) \right]
\]

analogous to the bond pricing formula if we regard \( \nu(t) \) as analogous to the instantaneous spot interest rate. Hence, we can “borrow” closed form
solutions for zero coupon bonds that arise under affine, quadratic term structure models, etc.
The General Case of Correlated Time Changes

- More generally, the generalized CF of $Y_t \equiv X_{T_t}$ under measure $P$ can be represented as the “Laplace transform” of $T_t$ under a new complex-valued measure $Q(\theta)$, evaluated at the characteristic exponent $\Psi_x(\theta)$ of $X_t$,

$$\phi_{Y_t}(\theta) \equiv E\left[e^{i\theta^\top Y_t}\right] = E^\theta\left[e^{-T_t\Psi_x(\theta)}\right] \equiv L^\theta_{T_t}(\Psi_x(\theta)).$$  \hspace{1cm} (1)

- For each $\theta \in \mathcal{D}$, $Q(\theta)$ is absolutely continuous with respect to $P$ and is defined by

$$E\frac{dQ(\theta)}{dP}\bigg|_{\mathcal{F}_{T_t}} \equiv M_t(\theta) \equiv \exp\left(i\theta^\top Y_t + T_t\Psi_x(\theta)\right), \quad \theta \in \mathcal{D},$$  \hspace{1cm} (2)

which is a complex valued $P$-martingale with respect to $\{\mathcal{F}_t|t \geq 0\}$, for each $\theta \in \mathcal{D}$. 
• Why is $M_t(\theta) \equiv E^{dQ(\theta)}_{\mathcal{F}_t} = \exp \left( i\theta^\top Y_t + T_t \Psi_x(\theta) \right), \theta \in \mathcal{D}$ a $P$ martingale?
  – Recall the familiar Wald martingale defined on a Lévy process
    \[ Z_t(\theta) \equiv \exp \left( i\theta^\top X_t + t \Psi_x(\theta) \right). \]
  – Time change (i.e. replacing $t$ by $T_t$) preserves the martingality.

• Theorem proof:
  \[
  E \left[ e^{i\theta^\top Y_t} \right] = E \left[ e^{i\theta^\top Y_t + T_t \Psi_x(\theta) - T_t \Psi_x(\theta)} \right] \\
  = E \left[ M_t(\theta) e^{-T_t \Psi_x(\theta)} \right] = E^\theta \left[ e^{-T_t \Psi_x(\theta)} \right] \equiv L_{T_t}^\theta (\Psi_x(\theta)).
  \]

• The complex-valued measure loses its probabilistic interpretation, but the mathematical operation remains valid.
• Under measure $Q(\theta)$, we can take “expectations” as if there is no correlation $\Rightarrow$ leverage-neutral measure.
Let $S_t$ be the time-$t$ price of a limited liability asset under statistical measure:

$$S_t \equiv S_0 e^{\vartheta^\top Y_t}, \quad t \geq 0,$$

for given $S_0 > 0$, where recall $Y_t \equiv X_{T_t}$.

The generalized CF of the log return $s_t \equiv \ln(S_t/S_0)$ is

$$\phi_s(\theta) \equiv E\left[e^{i\theta s_t}\right] = E\left[e^{i\theta \vartheta^\top Y_t}\right] = \mathcal{L}_T^{\theta \vartheta}(\Psi_x(\theta \vartheta)), \quad t \geq 0, \theta \vartheta \in \mathcal{D} \subseteq \mathbb{C}^d.$$

Let $F_t(M)$ be the $M$ maturity forward price at time $t \in [0, M]$. To value most European-style claims maturing at $M$, specify $F$ as a positive martingale

$$F_t(M) \equiv F_0(M) e^{\vartheta^\top Y_t + T_t^\top \Psi_x(-i\vartheta)}, \quad t \in [0, M]$$

under an $M$–forward measure. $T_t$ is now a vector of stochastic clocks so that $e^{\vartheta^\top Y_t + T_t^\top \Psi_x(-i\vartheta)}$ is an exponential martingale.
The generalized CF of the terminal log return $f_M \equiv \ln(F_M(M)/F_0(M))$ is
$$\phi_f(\theta) \equiv E[e^{i\theta f_M}] = \mathcal{L}_T^{\theta\vartheta}(\Psi_x(\theta\vartheta) - i\theta\Psi_x(-i\vartheta)).$$
## Specification Analysis Ia

### Lévy processes and Characteristic Exponents:

<table>
<thead>
<tr>
<th>Pure Continuous Lévy component</th>
<th>Finite Activity Pure Jump Lévy components</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu t + \sigma W_t$</td>
<td>$\lambda \frac{1}{\sqrt{2\pi}\sigma_j^2} \exp\left(-\frac{(x-\alpha)^2}{2\sigma_j^2}\right)$</td>
</tr>
<tr>
<td>$-i\mu \theta + \frac{1}{2}\sigma^2\theta^2$</td>
<td>$\lambda \left(1 - e^{i\theta\alpha - \frac{1}{2}\sigma_j^2\theta^2}\right)$</td>
</tr>
</tbody>
</table>

- **Merton (76)**
  - $\lambda \frac{1}{2\pi}\exp\left(-\frac{|x-k|}{\eta}\right)$
  - $\lambda \left(1 - e^{i\theta k \frac{1-\eta^2}{1+\theta^2\eta^2}}\right)$

- **Kou (99)**
  - $\lambda \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right)$
  - $\lambda \left(1 - \frac{1}{1-i\theta\eta}\right)$

- **Eraker (2001)**
  - $\lambda \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right)$
  - $\lambda \left(1 - \frac{1}{1-i\theta\eta}\right)$
**Specification Analysis Ib**

**Lévy processes and Characteristic Exponents:** *Infinite Activity Jumps*

<table>
<thead>
<tr>
<th>Lévy Components</th>
<th>Lévy Density $\Pi(dx)/dx$</th>
<th>Characteristic Exponent $\Psi(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIG</td>
<td>$e^{\beta x} \frac{\delta \alpha}{\pi</td>
<td>x</td>
</tr>
<tr>
<td>Hyperbolic</td>
<td>$e^{\beta x} \left[ \frac{\int_{0}^{\infty} y^{2y+\alpha^2</td>
<td>x</td>
</tr>
<tr>
<td>CGMY</td>
<td>$\begin{cases} Ce^{-G</td>
<td>x</td>
</tr>
<tr>
<td>VG</td>
<td>$\frac{\mu^2_{\pm} \exp\left(\frac{-\mu_{\pm}}{v_{\pm}}</td>
<td>x</td>
</tr>
<tr>
<td>LS</td>
<td>$c</td>
<td>x</td>
</tr>
</tbody>
</table>

\[ (\mu_{\pm} = \sqrt{\frac{\alpha^2}{4\lambda^2} + \frac{\sigma^2}{2\lambda} \pm \frac{\alpha}{2\lambda}}, v_{\pm} = \mu^2_{\pm}/\lambda) \]
### Specification Analysis IIa

**Activity Rate Dynamics and the Laplace Transform**

*Affine: Duffie, Pan, Singleton (2000)*

<table>
<thead>
<tr>
<th>Activity Rate Specification $v(t)$</th>
<th>Laplace Transform $L_{T_t}(\lambda) \equiv E \left[ e^{-\lambda T_t} \right]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t) = b_v^\top Z_t + c_v,$</td>
<td>$\exp \left( -b(t)^\top z_0 - c(t) \right),$</td>
</tr>
<tr>
<td>$\mu(Z_t) = a - \kappa Z_t,$</td>
<td>$b'(t) = \lambda b_v - \kappa^\top b(t) - \frac{1}{2} \beta b(t)^2$</td>
</tr>
<tr>
<td>$[\sigma(Z_t)\sigma(Z_t)^\top]_{ij} = \alpha_i + \beta_i^\top Z_t,$</td>
<td>$-b_{\gamma} \left( L_q(b(t)) - 1 \right),$</td>
</tr>
<tr>
<td>$[\sigma(Z_t)\sigma(Z_t)^\top]_{ij} = 0, \quad i \neq j,$</td>
<td>$c'(t) = \lambda c_v + b(t)^\top a - \frac{1}{2} b(t)^\top \alpha b(t)$</td>
</tr>
<tr>
<td>$\gamma(Z_t) = a_{\gamma} + b_{\gamma}^\top Z_t.$</td>
<td>$-a_{\gamma} \left( L_q(b(t)) - 1 \right),$</td>
</tr>
</tbody>
</table>

$b(0) = 0, \ c(0) = 0.$
Activity Rate Dynamics and the Laplace Transform:
Generalized Affine: Filipovic (2001)

<table>
<thead>
<tr>
<th>Activity Rate Specification $v(t)$</th>
<th>Laplace Transform $\mathcal{L}_{T_i}(\lambda) \equiv E \left[ e^{-\lambda T_i} \right]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A} f(x) = \frac{1}{2} \sigma^2 x f''(x) + (a' - \kappa x) f'(x)$ + $\int_{\mathbb{R}_0^+} (f(x+y) - f(x) + f'(x) (1 \land y))$ $(m(dy) + x \mu(dy)),$</td>
<td>$\exp(-b(t)v_0 - c(t)),$</td>
</tr>
<tr>
<td>$a' = a + \int_{\mathbb{R}_0^+} (1 \land y) m(dy),$</td>
<td>$b'(t) = \lambda - \kappa b(t) - \frac{1}{2} \sigma^2 b(t)^2$ + $\int_{\mathbb{R}_0^+} \left( 1 - e^{-y^2(t)} - b(t) (1 \land y) \right) \mu(dy),$</td>
</tr>
<tr>
<td>$\int_{\mathbb{R}_0^+} \left[ (1 \land y) m(dy) + (1 \land y^2) \mu(dy) \right] &lt; \infty.$</td>
<td>$c'(t) = ab(t) + \int_{\mathbb{R}_0^+} \left( 1 - e^{-y^2(t)} \right) m(dy),$</td>
</tr>
<tr>
<td></td>
<td>$b(0) = c(0) = 0.$</td>
</tr>
</tbody>
</table>
## Specification Analysis IIc

### Activity Rate Dynamics and the Laplace Transform

*Quadratic: Leippold and Wu (2002)*

<table>
<thead>
<tr>
<th>Activity Rate Specification</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>$\mathcal{L}_{T_t}(\lambda) \equiv E \left[ e^{-\lambda T_t} \right]$</td>
</tr>
</tbody>
</table>

$$
\begin{align*}
\mu(Z) &= -\kappa Z, \quad \sigma(Z) = I, \\
v(t) &= Z_t^T A_v Z_t + b_v^T Z_t + c_v.
\end{align*}
$$

$$
\begin{align*}
A'(t) &= \lambda A_v - A(t) \kappa - \kappa^T A(\tau) - 2A(t)^2, \\
b'(t) &= \lambda b_v - \kappa b(t) - 2A(t)^T b(t), \\
c'(t) &= \lambda c_v + tr A(t) - b(t)^T b(t)/2, \\
A(0) &= 0, b(0) = 0, c(0) = 0.
\end{align*}
$$

$$
\begin{align*}
\exp \left[ -z_0^T A(t) z_0 - b(t)^T z_0 - c(t) \right],
\end{align*}
$$
Correlations and Measure Changes

• Correlation via diffusions: Example

\[ X_t = W_t, \quad dv(t) = (a - \kappa v(t))dt + \eta \sqrt{v(t)}dZ_t, \quad dW_t dZ_t = \rho dt. \]

– Measure change:

\[ \frac{dQ(\theta)}{dP} \bigg|_{\mathcal{F}_t} = \exp \left( i \theta Y_t + \frac{1}{2} \theta^2 \int_0^t v(s)ds \right). \]

– \( v(t) \) dynamics under \( Q(\theta) \): 

\[ dv(t) = (a - (\kappa - i \theta \eta \rho)v(t))dt + \eta \sqrt{v(t)}dZ_t, \]

which remains affine.
Specification Analysis IIIa

Correlations and Measure Changes: Correlation via jumps:

- Example
  \[ X_t = L_{t-1}^\alpha, dv(t) = (a - \kappa v(t)) dt - \beta t^{1/\alpha} dL_{t-1}^\alpha, \]
  where \( L_{t-1}^\alpha \) denotes a standard Lévy \( \alpha \)-stable motion with tail index \( \alpha \in (1, 2] \) and maximum negative skewness.

- Measure change:
  \[ \frac{dQ(\theta)}{dP} \bigg|_{\mathcal{F}_t} = \exp \left( i\theta L_{t-1}^\alpha + \Psi_x(\theta) T_t \right), \quad \Psi_x(\theta) = - (i\theta)^\alpha \sec \frac{\pi \alpha}{2}, \quad \text{Im} (\theta) < 0. \]

- Under this new (leverage-neutral) measure, \( \Pi^\theta(dx) = e^{i\theta x} \Pi(dx) \), and \( v(t) \) satisfies generalized affine:
  \[ L_{t-1}^\theta (\Psi_x(\theta)) = \exp (-b(t)v_0 - c(t)). \quad (3) \]
Pricing State Contingent Claims

• Consider a payoff at a given fixed time $M$ which is any linear combination of the following payoffs:

$$\Pi_Y(k; a, b, \vartheta, c) = \left( a + be^{\vartheta^\top Y_M} \right) 1_{c^\top Y_M \leq k}$$

• Examples of claims covered by the above structure include:
  – European call with strike $K$: $\Pi(\ln(F_0(M)/K); -K, F_0(M), \vartheta, -\vartheta)$,
  – European put with strike $K$: $\Pi(\ln(K/F_0(M)); K, -F_0(M), \vartheta, \vartheta)$,
  – A protected put: $\max[S_M, K] = \Pi(\ln(F_0(M)/K); 0, F_0(M), \vartheta, -\vartheta) + \Pi(\ln(K/F_0(M)); K, 0, \vartheta, \vartheta)$,
  – A binary call: $\Pi(\ln(F_0(M)/K); 1, 0, 0, -\vartheta)$.

where recall that $F_t(M) = F_0(M)e^{\vartheta^\top Y_t}$ is the $M$ maturity forward price of the underlying asset at time $t \in [0, M]$.

• State price: Let $G(k; a, b, \vartheta, c; M)$ denote the price of such a claim. We can compute $G$ with two transform methods using $\phi_Y$. 
Transform I

• Let $G^I(z; a, b, \vartheta, c)$ denote a Fourier transform of state price $G(k; a, b, \vartheta, c)$, defined as

$$G^I(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{+\infty} e^{izk} dG(k; a, b, \vartheta, c), \quad z \in \mathbb{R}. \quad (4)$$

• $G^I(z; a, b, \vartheta, c)$ can be written as an affine function of the generalized Fourier transform of $Y_M$:

$$G^I(z; a, b, \vartheta, c) = a \phi_Y(zc) + b \phi_Y(zc - i\vartheta).$$

• The price $G(k; a, b, \vartheta, c)$ can then be obtained by inversion:

$$G(k; a, b, \vartheta, c) = \frac{G^I_0}{2} + \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{izk} G^I(-z; a, b, \vartheta, c) - e^{-izk} G^I(z; a, b, \vartheta, c)}{iz} dz,$$

where $G^I_0 = G^I(0; a, b, \vartheta, c) = a + b \phi_Y(-i\vartheta)$.

Note that this is a one-dimensional inversion regardless of the dimensionality of the uncertainty $Y_M$. 

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Transform II

- Define a second transform $G^{II}(z; a, b, \vartheta, c)$:

$$
G^{II}(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{+\infty} e^{izk} G(k; a, b, \vartheta, c) \, dk, \quad z \in \mathbb{C} \subseteq \mathbb{C}.
$$

$$
G^{I}(z; a, b, \vartheta, c) \equiv \int_{-\infty}^{+\infty} e^{izk} dG(k; a, b, \vartheta, c), \quad z \in \mathbb{R}.
$$

Note the two differences between $G^{I}$ and $G^{II}$.

- $G^{II}$, when well-defined, is given by:

$$
G^{II}(z; a, b, \vartheta, c) = \frac{i}{z} (a \phi_Y(zc) + b \phi_Y(zc - i\vartheta)).
$$

- Inversion:

$$
G(k) = \frac{1}{2\pi} \int_{iz_{-\infty}}^{iz_{+\infty}} e^{-izk} G^{II}(z; a, b, \vartheta, c) \, dz.
$$

- This inversion can be performed numerically using FFT or Fractional FFT, generating superior computational efficiency. By vectorizing in maturity, the model’s option values at all strikes and maturities can be obtained in one stroke.
The choice of $\text{Im } z$ is crucial and depends upon the payoff structure.

<table>
<thead>
<tr>
<th>Contingent Claim</th>
<th>Generalized transform $-iz\varphi(z)$</th>
<th>Restrictions on $\text{Im } z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(k; a, b, \vartheta, c)$</td>
<td>$a\phi_Y(zc) + b\phi_Y(zc - i\vartheta)$</td>
<td>$(0, \infty)$</td>
</tr>
<tr>
<td>$G(-k; a, b, \vartheta, c)$</td>
<td>$a\phi_Y(-zc) + b\phi_Y(-zc - i\vartheta)$</td>
<td>$(-\infty, 0)$</td>
</tr>
<tr>
<td>$e^{\alpha k}G(k; a, b, \vartheta, c)$</td>
<td>$a\phi_Y((z - i\alpha)c) + b\phi_Y((z - i\alpha)c - i\vartheta)$</td>
<td>$(\alpha, \infty)$</td>
</tr>
<tr>
<td>$e^{\beta k}G(-k; a, b, \vartheta, c)$</td>
<td>$a\phi_Y(-(z - i\beta)c) + b\phi_Y(-(z - i\beta)c - i\vartheta)$</td>
<td>$(-\infty, \beta)$</td>
</tr>
<tr>
<td>$e^{\alpha k}G(k; a_1, b_1, \vartheta_1, c_1) + e^{\beta k}G(-k; a_2, b_2, \vartheta_2, c_2)$</td>
<td>$a_1\phi_Y((z - i\alpha)c_1) + b_1\phi_Y((z - i\alpha)c_1 - i\vartheta_1) + a_2\phi_Y(-(z - i\beta)c_2) + b_2\phi_Y(-(z - i\beta)c_2 - i\vartheta_2)$</td>
<td>$(\alpha, \beta)$</td>
</tr>
</tbody>
</table>

($\alpha, \beta, a, b$ are real constants with $\alpha < \beta$.)
Summary

- We provide a powerful tool in generating tractable option pricing models.
  - Apply stochastic time change to Lévy processes
  - Prove a theorem that facilitates the derivation of the CF.
  - Price options via transform methods.

- Applications: Model design and calibration
  - Huang and Wu (JF 2004): Specification analysis for equity index options
  - Carr and Wu (wp, 2004): Stochastic skew in currency options
  - Bakshi, Carr, and Wu (wp, 2004): Stochastic discount factors in international economies.
Approach #3: Time-Changed Markov Processes

- Consider a time homogeneous Markov process $X$ with infinitessimal generator $\mathcal{G}$.
- Formally, the forward price at time $t$, maturity $T$ of a contingent claim paying $f(X_T)$ at $T$ is given by:

$$ E^Q[f(X_T)|X_t = x] = [e^{(T-t)\mathcal{G}} f](x). $$

- Now let $\phi(x)$ be an eigenfunction solving:

$$ \mathcal{G}\phi(x) = \lambda \phi(x). $$

- The forward value at time $t$ of a claim with an eigenfunction payoff $\phi(X_T)$ paid at $T$ factors into the product of its intrinsic value $\phi(X_t)$ and its time value $e^{(T-t)\lambda}$:

$$ E^Q[f(X_T)|X_t = x] = \phi(x)e^{(T-t)\lambda}. $$
Let \( \tau_t \) be a stochastic clock, i.e. an increasing random process with \( \tau_0 = 0 \).

Suppose that the random clock \( \tau \) evolves independently of the time-homogeneous Markov process \( X \). Let \( Y_t \) be the process arising if \( X \) is run on \( \tau \):

\[ Y_t \equiv X_{\tau_t}, \quad t \geq 0. \]

Recall that \( \phi(x) \) is an eigenfunction of the generator \( G \) of \( X \), i.e. \( \phi(x) \) solves \( G\phi(x) = \lambda \phi(x) \).

Using a conditioning argument, it can be shown that the forward value at time \( t \) of a claim with an eigenfunction payoff \( \phi(Y_T) \) paid at \( T \) factors into the product of its intrinsic value \( \phi(Y_t) \) and its time value \( E^Q e^{\lambda (\tau_T - \tau_t)} \):

\[ E^Q[f(Y_T)|Y_t = y] = \phi(y) E^Q e^{\lambda (\tau_T - \tau_t)}. \]

Thus, if the Laplace transform of \( \tau_T - \tau_t \) is known in closed form, then one can value claims with eigenfunction payoffs. As the eigenfunctions form a basis, one can also value arbitrary European claims by integrating/summing across eigenvalues.
Appendix 1: Elimination of the Market Price of Volatility Risk

One can derive the following PDE governing the function $C^{(1)}(t, S, Y)$ relating the price of an option to time $t$, the price of the underlying stock $S$, and the price of a state variable governing volatility $Y$:

$$\frac{\partial C^{(1)}}{\partial t} + r \left[ S \frac{\partial C^{(1)}}{\partial S} - C^{(1)} \right] + \left[ \alpha(m - Y) - \lambda \beta \right] \frac{\partial C^{(1)}}{\partial Y} + \frac{f^2(Y)}{2} \frac{\partial^2 C^{(1)}}{\partial S^2} + \rho f(Y) \beta \frac{\partial^2 C^{(1)}}{\partial S \partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(1)}}{\partial Y^2} = 0,$$

where $\lambda$ is the market price of volatility risk. The above PDE also holds for the price of a second option:

$$\frac{\partial C^{(2)}}{\partial t} + r \left[ S \frac{\partial C^{(2)}}{\partial S} - C^{(2)} \right] + \left[ \alpha(m - Y) - \lambda \beta \right] \frac{\partial C^{(2)}}{\partial Y} + \frac{f^2(Y)}{2} \frac{\partial^2 C^{(2)}}{\partial S^2} + \rho f(Y) \beta \frac{\partial^2 C^{(2)}}{\partial S \partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(2)}}{\partial Y^2} = 0.$$

Let $\gamma(t, S, C^{(2)})$ be the function relating the price of the first option to time $t$, the underlying stock price $S$, and the price of the second option:

$$\gamma(t, S, C^{(2)}) \equiv C^{(1)}(t, S, Y),$$

where $C^{(2)}$ solves (6).

Equivalently:

$$C^{(1)}(t, S, Y) = \gamma(t, S, C^{(2)}(t, S, Y)) \quad (7)$$

Differentiating once:

$$\frac{\partial C^{(1)}}{\partial t} = \gamma_1 + \gamma_3 \frac{\partial C^{(2)}}{\partial t} \quad (8)$$

$$\frac{\partial C^{(1)}}{\partial S} = \gamma_2 + \gamma_3 \frac{\partial C^{(2)}}{\partial S} \quad (9)$$

$$\frac{\partial C^{(1)}}{\partial Y} = \gamma_3 \frac{\partial C^{(2)}}{\partial Y} \quad (10)$$

Differentiating one more time:

$$\frac{\partial^2 C^{(1)}}{\partial S^2} = \gamma_{22} + 2 \gamma_{23} \frac{\partial C^{(2)}}{\partial S} + \gamma_{33} \left( \frac{\partial C^{(2)}}{\partial S} \right)^2 + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial S \partial Y} \quad (11)$$
\[
\frac{\partial^2 C^{(1)}}{\partial S \partial Y} = \gamma_{23} \frac{\partial C^{(2)}}{\partial Y} + \gamma_{33} \frac{\partial C^{(2)}}{\partial S} \frac{\partial C^{(2)}}{\partial Y} + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial S \partial Y} \tag{12}
\]

\[
\frac{\partial^2 C^{(1)}}{\partial Y^2} = \gamma_{33} \left( \frac{\partial C^{(2)}}{\partial Y} \right)^2 + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial Y^2} \tag{13}
\]

Substituting (8) to (13) in (5):

\[
\gamma_1 + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial Y^2} + r[S[\gamma_2 + \gamma_3 \frac{\partial C^{(2)}}{\partial S}] - \gamma_1] + [\alpha(m - Y) - \lambda \beta] \frac{\partial C^{(2)}}{\partial Y} \\
+ \frac{f^2(Y)}{2} \left[ \gamma_{22} + 2 \gamma_{23} \frac{\partial C^{(2)}}{\partial S} + \gamma_{33} \left( \frac{\partial C^{(2)}}{\partial S} \right)^2 + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial S^2} \right] + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial Y} + \gamma_3 \frac{\partial C^{(2)}}{\partial S} \frac{\partial C^{(2)}}{\partial Y} + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial S \partial Y} \\
+ \frac{\beta^2}{2} \left( \gamma_{23} \left( \frac{\partial C^{(2)}}{\partial Y} \right)^2 + \gamma_3 \frac{\partial^2 C^{(2)}}{\partial Y^2} \right) = 0.
\]

Re-arranging terms:

\[
\gamma_1 + r[S \gamma_2 - \gamma] + \gamma_3 \left[ \frac{\partial C^{(2)}}{\partial a} + rS \frac{\partial C^{(2)}}{\partial S} + \frac{(\alpha(m - Y) - \lambda \beta) \partial C^{(2)}}{\partial Y} + \frac{f^2(Y)}{2} \frac{\partial^2 C^{(2)}}{\partial S^2} + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(2)}}{\partial Y^2} \right] \\
+ \frac{f^2(Y) \gamma_{22}}{2} + \gamma_{23} \left[ \frac{f^2(Y) \frac{\partial C^{(2)}}{\partial S}}{\partial a} + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(2)}}{\partial Y^2} \right] + \gamma_{33} \left[ \frac{f^2(Y)}{2} \left( \frac{\partial C^{(2)}}{\partial S} \right)^2 + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2} \left( \frac{\partial C^{(2)}}{\partial Y} \right)^2 \right] = 0.
\]

Substituting (6) into the above simplifies the result to:

\[
\gamma_1 + r[S \gamma_2 - \gamma] + r C^{(2)} \gamma_3 \\
+ \frac{f^2(Y) \gamma_{22}}{2} + \gamma_{23} \left[ \frac{f^2(Y) \frac{\partial C^{(2)}}{\partial S}}{\partial a} + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2} \frac{\partial^2 C^{(2)}}{\partial Y^2} \right] + \gamma_{33} \left[ \frac{f^2(Y)}{2} \left( \frac{\partial C^{(2)}}{\partial S} \right)^2 + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial Y} + \frac{\beta^2}{2} \left( \frac{\partial C^{(2)}}{\partial Y} \right)^2 \right] = 0.
\]

Now the risk-neutral dynamics of $S$ are:

\[
dS_t = rS_t dt + f(Y_t) dW_t.
\]

Note that the variance rate of $S$ is:

\[
\text{Var}(dS) = f^2(Y_t) dt.
\]

The risk-neutral dynamics of $C^{(2)}$ are given by:

\[
dC^{(2)} = rC^{(2)} dt + \frac{\partial C^{(2)}}{\partial S} f(Y_t) dW_t + \frac{\partial C^{(2)}}{\partial Y} \beta dW_t,
\]

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where $dW_1dW_2 = \rho dt$. Note that the covariance rate of $C^{(2)}$ with $S$ is:

$$\text{Cov}(dC^{(2)}, dS) = \left[ f^2(Y) \frac{\partial C^{(2)}}{\partial S} + \rho f(Y) \beta \frac{\partial C^{(2)}}{\partial S} \frac{\partial C^{(2)}}{\partial Y} \right] dt,$$

while the variance rate of $C^{(2)}$ is:

$$\text{Var}(dC^{(2)}) = \left[ f^2(Y) \left( \frac{\partial C^{(2)}}{\partial S} \right)^2 + 2\rho f(Y) \beta \frac{\partial C^{(2)}}{\partial S} \frac{\partial C^{(2)}}{\partial Y} + \beta^2 \left( \frac{\partial C^{(2)}}{\partial Y} \right)^2 \right] dt.$$

Substituting into the above PDE gives:

$$\gamma_1 + r[S\gamma_2 - \gamma] + rC^{(2)}\gamma_3 + \frac{1}{2} \frac{\text{Var}(dS)}{dt} \gamma_22 + \frac{\text{Cov}(dC^{(2)}, dS)}{dt} \gamma_23 + \frac{1}{2} \frac{\text{Var}(dC^{(2)})}{dt} \gamma_33 = 0. \quad (14)$$

If one can exogenously model the volatility structure of $C^{(2)}$ without reference to $\lambda$, then one does not need to specify $\lambda$. There are many ways to do this.