Home-grown Diversification with Volatility Products

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This Version: September 11, 2014

ABSTRACT

Recent changes to clearing-house regulations have promoted exchange-traded products offering profiles previously accessible only over-the-counter. Thus, as correlations grow between international markets for traditional assets, a new strand of portfolio diversification research questions the benefits of home-grown diversification using volatility futures, or exchange-traded products that are based on these contracts. The focus of our paper is the question: "How high should the expected return on volatility be so that investors in equity and bonds perceive, ex-ante, that diversification into volatility is optimal?" This *optimal diversification threshold* is derived under three standard theoretical optimization frameworks. An empirical study of US and European markets shows that diversification into short-term volatility futures is frequently perceived to be optimal, ex-ante, whatever the optimization model, parameters, and risk preference. However, ex-post analysis shows that these perceived diversification benefits are rarely realized, being eroded by an unusually high roll cost.

JEL classification: G11, G15, G23

Keywords: Black–Litterman model, Mean–variance criterion, Skewness preference, VIX futures, Volatility ETNs

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1 Introduction

The debate surrounding the benefits of international portfolio diversification has heightened since the banking crisis as international equities and bonds and commodities become more highly correlated. This has been accompanied by substantial developments in the home-grown investment universe, including alternative investments such as real estate or hedge funds. The combined effect has been to precipitate a renewed search for alternative domestic diversifiers. In particular, equity volatility, celebrated as a highly innovative asset class, arises as a natural diversification choice because its negative correlation with equity increases exactly when diversification is needed most, a fact that has been well documented in the literature. For example, during the financial crisis 2008–2010, the negative correlation between the S&P 500 and its corresponding volatility index VIX remained at extraordinary high levels of -0.85. Thus if on April 1, 2010 an S&P 500 investor had put 30% of his capital in the risk-free asset and taken an equivalent long position in the June 2010 VIX futures contract, closing the position a week before expiry, he would have achieved an annualized Sharpe ratio of 3.61. Holding the SPY alone gave a negative mean excess return over the same period.

These observations motivate the question whether volatility could be an effective diversification tool for pension funds, public companies and indeed any investor that is long in domestic capital assets, i.e. equities and bonds. Certain investment entities such as mutual funds and pension funds are forbidden by law to short equity, because this is generally considered as speculation and not as a position that fits a long-term investment. For this reason we consider an investor who holds a domestic long equity-only or equity-bond portfolio and seeks to diversify into volatility.

Over-the-counter (OTC) trades such as variance swaps are also disallowed for many investors. Having said this, recent regulations have generally moved demand away from OTC transactions and towards exchange-listed products instead. The Dodd-Frank Act in the U.S. and the European Union EMIR regulations require OTC transactions to be cleared by central counterparties in much the same way as exchange-traded products. This has acted as a catalyst for growth in listed products such as futures, notes and funds which attempt to mimic the risk-return characteristics of popular OTC instruments. So for instance, from 2009 to 2013, the number of traded VIX futures contracts increased from 4,500 to more than 150,000 contracts with a value of more than $2.8bn traded on average each day. Accordingly, the market for exchange-traded products based on volatility futures has exploded in recent years and trading volume on some of these products can reach about $5bn per day (see Alexander et al. (2015) for further details).

This paper begins by introducing a new theoretical concept. Given an investor that has a long position on each of the assets or financial instruments $X_1, X_2, ..., X_{k-1}$, the *optimal diversification threshold* for the asset/instrument X_k is the lowest expected return q_k on X_k for which an additional long position on X_k is perceived to be optimal, ex-ante. It is interesting to apply this concept over a substantial historical period with regular periodic rebalancing in order to identify the frequency with which a 'rational' investor (i.e. one following a portfolio optimization model) would perceive diversification to be ex-ante optimal. If the expected returns have never been high enough to perceive that diversification is ex-ante optimal, then the same investor should ignore marketing statements that X_k could be a valuable diversification tool.

We shall derive a general expression for the optimal diversification threshold in the context of three standard optimization frameworks and apply it to the problem of diversification with volatility futures for an investor that has long positions in equity-only or in equity and bonds. In general, both the threshold and the corresponding optimal diversification frequency will depend on the optimization model used by the investor (e.g. mean-variance, or more general utility maximization) and the parameters of the model (e.g. the covariance matrix of the k assets) and the risk preferences of the investor.

Our theoretical results are then used to analyze the perceived benefits of volatility diversification for long equity or equity-bond investors in the U.S. and European Union markets. Using data from 2006 to 2013 we investigate how often diversification into long positions in volatility futures has been perceived as optimal, ex-ante. Then we compare the realized performance of the optimally-diversified portfolios with that of traditional equity-bond portfolios.

Our work makes some significant contributions to the existing literature in this area. Most of the academic papers that advocate the use of volatility as an effective diversifier have studied variance swaps, or data on the VIX index as an indicative quote for the variance swap rate, as in Dash and Moran (2005) or Daigler and Rossi (2006). But variance swap rates and the VIX have very different statistical characteristics to VIX futures, as shown by many authors. Relatively few previous studies have examined equity diversification using volatility futures and most of these have employed an ex-post analysis based on ad-hoc allocations with samples that focus on the turbulent period covering the credit and banking crises from 2007 to 2009. Our empirical study is the first that applies an ex-ante analysis within a rolling framework, i.e. a situation where the investor periodically rebalances his portfolio based on new information. We also use a much longer sample period than any of the previous studies on volatility diversification.

In the following: Section 2 motivates our work by setting it in the context of the relevant literature on diversification; Section 3 presents our theoretical results and applies them to the problem of volatility diversification for (i) a mean-variance investor, (ii) an investor with a more general utility function which includes aversion to negative skew, and (iii) an investor using the framework of Black and Litterman (1992); Section 4 presents our empirical results and Section 5 concludes.

2 Literature Review

One of the earliest and most influential papers on the benefits of home-made diversification is Errunza, Hogan, and Hung (1999). Using monthly data between 1976 and 1993 for seven developed and nine emerging markets they use return correlations, mean–variance spanning and change in Sharpe ratio tests as evidence that, once the investor has employed home-grown diversification tools, gains from international diversification are statistically and economically insignificant. Since then several studies have shown that correlations between international stock markets are particularly high during times of market stress, which lowers international diversification benefits at a time when it is needed most. For example, Butler and Joaquin (2002) measure the correlations of US, UK, Japanese, Australian and European stock market indices between January 1970 to December 2000, observing a non-normal behavior of returns correlations with significantly higher correlations in bear markets than in calm or bull markets. Kearney and Lucey (2004) survey previous literature on international equity market integration and provide further evidence of declining diversification benefits in international equity markets due to increasing correlations. More recently, Liu et al. (2014) explore ten European equity markets from 2001 to March 2013. They construct three optimal portfolios for the PIIGS (Portugal, Italy, Ireland, Greece and Spain), CORE (Austria, Finland, France, Germany and the Netherlands) and all ten countries, and demonstrate that there are only limited diversification benefits within the euro-zone as the optimal portfolio in each group consists mainly of one index. Other recent evidence suggests that international equity investors are now seeking less correlated markets. Using daily data for the main stock markets indices during 2001–2009, Vermeulen (2013) finds a significant negative relationship between foreign equity holdings and stock markets correlations during the recent financial crisis. This stresses the importance of finding new markets for effective diversification, as advocated by Coerdacier and Guibaud (2011).

Furthermore, several studies attest to the limited diversification potential of commodities. Cheung and Miu (2010) show that the diversification benefits of adding commodity futures concentrate in bullish commodity markets, which are rather rare events (between 1970 and 2005 they identified only two such periods). During bearish markets commodities offer weak, if any diversification benefits. In a similar study, Rudolf et al. (1993) investigate the regime-switching behavior of equity–commodities correlations. Using the example of the S&P Goldman Sachs commodity index GSCI from April 1970 to April 1991, the authors show that correlations rise especially during periods of market stress, thus lowering diversification benefits just when they are most important. Simple calculations with recent data support these empirical findings: for instance, the sample correlation between the daily returns on the S&P 500 stock index and those on the GSCI was only 0.17 (January 2004 to December 2008) but then rose to 0.55 (January 2009 to June 2013).

Daskalaki and Skiadopoulos (2011) consider a portfolio allocation setting where the investor allocates funds between equities, bonds, the risk-free asset and commodities. They consider both an in-sample and an out-of-sample setting and also take the higher moments of the asset returns distribution into account. They also apply various utility functions that describe the preferences of the investor. A rich data set is employed that covers the period between January 1989 and December 2009. Overall, diversification benefits from the inclusion of commodities are only identified during the 2005–2008 commodity boom period. Results against diversification benefits of commodities hold irrespective of the performance measure employed, the utility function specified or the particular commodity used (with the exception of gold).

Finally, there is little evidence that alternative asset classes, such as real estate or funds of funds, can complement traditional capital asset portfolios – see Mull and Soenen (1997) and Gueyie and Amvella (2006) for further details. More recently, Kroencke and Schindler (2012) show that although international real estate can contribute to portfolio diversification in general, their diversification benefits were statistically and economically insignificant during the financial crisis.

Turning now to the growing literature on volatility diversification, several papers that advocate volatility as an effective diversifier have studied variance swaps. For instance, Hafner and Wallmeier (2008) focus on European variance swaps and Egloff et al. (2010) consider US equity investors. In a slightly different vein, Copeland and Copeland (1999) use the percentage change between today's VIX and its 75-day moving average as a signal to switch between value and growth portfolios or between large-cap and small-cap strategies. By applying different modifications of this trading rule (i.e. various levels for the change in VIX and different holding periods), they find that the vast majority of cases produces positive returns for the investor.

Work on diversification using volatility futures is more limited and much less successful in adequately demonstrating its benefits. Using only ex-post analysis Hill (2013) confirms the recommendation of Whaley (2000) that VIX mid-term futures are useful diversification instruments for long-term investors. Other studies simply apply ad-hoc allocations to volatility and other asset classes and examine how such allocations performed empirically. For instance, following the ex-post analysis of VIX futures by Szado (2009), Stanescu and Tunaru (2012) use an ex-post empirical analysis to demonstrate the potential for diversification of equity and equity-bond exposure using VSTOXX futures, suggesting that they have higher diversification benefits than VIX futures. Similarly, Guobuzaite and Martellini (2012) confirm previous findings that mid-term futures are more suitable for diversification than short-term futures, this time for European markets. Warren (2012) examines the diversification benefits of adding long or short VIX futures positions to a base portfolio that includes US equity, fixed income and real estate exposure, finding that a short position in the prompt VIX futures enhances its Sharpe ratio. Again the empirical design is limited to an in-sample analysis but the data set covers a wider period than many previous studies.

The first paper to apply any optimization method to determine diversification allocations was Brière et al. (2010), who find diversification benefits for long equity investors within the minimum-variance optimization, based on an in-sample analysis with data ending in 2008. More specifically, they find that the inclusion of a long VIX futures position or a combination of long VIX futures and variance swaps positions both increase the Sharpe ratio of an equity-only portfolio in-sample (February 1990–July 1999). The conclusion still holds when the authors apply the in-sample optimized weights to an out-of-sample period (August 1999 - August 2008). Applying a similar methodology, Brière et al. (2012) consider an European equity investor who has the choice of investing in VIX or VSTOXX futures. The optimal portfolio is determined by minimizing the modified conditional Value-at-Risk which takes higher-order moments into account, and volatility diversified portfolios are found to have significantly lower risk and higher returns than the equity-only portfolio. The authors do not employ an out-of-sample analysis and the sample ends in 2010. Chen et al. (2011) use a mean–variance approach to add VIX futures to four base Fama and French (1992, 1993) US stock portfolios. Again, only an in-sample analysis is presented and the sample ends in 2008.

More recent work is presented by Hancock (2013), who uses four different hedging methodologies to determine the number of short-term VIX futures contracts to add to a S&P 500 portfolio. However, the results are highly sensitive to the portfolio strategy used and require a careful choice of the appropriate optimization tool. Also, her study focuses on hedging equity risk with volatility futures, rather than portfolio diversification. Although Stanton (2011) describes long equity investors as being implicitly short in volatility and thus considers long volatility as a hedge rather than a diversifier, hedging equity with its own futures may be more effective and less costly than buying volatility futures.

In summary, our paper fills an important gap in the literature. Unlike the work on variance swaps, it is relevant for mutual funds, pension funds and other long-term investors that cannot enter OTC contracts but seek new sources of diversification, especially during bear markets when other types of diversification fail. Within the listed-product literature, our study is first to use a rigorous ex-ante optimization framework. Moreover, we introduce a useful new concept (optimal diversification threshold) to the diversification literature and apply it to gain new and important insights to the diversification potential of volatility futures and their associated listed products.

3 Ex-Ante Optimal Volatility Diversification

We first consider the standard mean-variance (MV) framework introduced by Markowitz (1952). Then we extend this for negative skewness aversion to capture the typical features of equity returns during periods of market turmoil. Finally, we incorporate the effect of personal views and equilibrium returns as advocated by Black and Litterman (1992). In each case it facilitates the exposition to start with an analysis of the conditions under which it would be ex-post optimal for long equity investors to diversify into volatility.

3.1 Mean-Variance Optimality

The allocation of funds between risky assets (in this case equity, bonds and volatility ETNs) and a risk-free asset may be considered in two stages: (i) find all MV efficient combinations of risky assets; and (ii) find the optimal mix of one of these portfolios with the risk-free asset. The portfolio chosen from the stage (i) analysis is the tangency portfolio that when connected with the risk-free asset yields a linear efficient frontier with slope equal to the maximized Sharpe ratio (SR); the optimal choice along this frontier in stage (ii) is the portfolio that maximizes investor's expected utility. In stage (i) we need not consider specific risk preferences because we are only concerned with the convex frontier in {expected return, standard deviation} space.

In an ex-post analysis we seek to identify the times and conditions in the past under which a MV optimal allocation that was positive to equity and bonds was also positive to volatility.

This problem may be stated mathematically as:

$$\max_{w} SR = \frac{\mathbf{w^T q}}{\sqrt{\mathbf{w^T \Sigma w}}}, \tag{1}$$

where $\mathbf{w}=(w_s, w_b, w_v)^T$ is the vector of portfolio weights in equity, bonds and volatility assets respectively, with $w_s + w_b + w_v = 1$, $w_s > 0$, $w_b > 0$, $w_v \geq 0$, $\mathbf{q}$ is the vector of mean excess returns and $\mathbf{\Sigma}$ is their covariance matrix, both estimated ex-post using historical data.

Turning now to the ex-ante problem let us first consider the case of a long equity-only investor who may choose to finance his position by selling a discount bond. To allocate between equity, volatility and a discount bond according to the mean-variance (MV) criterion, this investor uses the unconstrained solution which maximizes the certainty equivalent (CE)

$$\mathbf{w^{mv}} = \gamma^{\mathbf{-1}} \mathbf{\Sigma^{-1} q}, \tag{2}$$

where $\mathbf{w^{mv}} = (w_s^{mv}, w_v^{mv})^T$ describes the allocation to the risky assets (w_s^{mv} and w_v^{mv} are not constrained to sum to 1, the allocation being completed with the residual invested in the risk-free asset); γ denotes the investor's coefficient of risk aversion; $\mathbf{q} = (q_s, q_v)^T$ is the vector of expected excess returns; $\mathbf{\Sigma}$ denotes their covariance matrix with elements σ_s^2, σ_v^2 and σ_{sv} and $\mathbf{\Sigma^{-1}}$ its inverse matrix. The solution may be rewritten:

$$w_s^{mv} = \gamma^{-1} \, |\Sigma|^{-1} (\sigma_v^2 q_s - \sigma_{sv} q_v), \quad w_v^{mv} = \gamma^{-1} \, |\Sigma|^{-1} (\sigma_s^2 q_v - \sigma_{sv} q_s), \tag{3}$$

where $|\Sigma| = (\sigma_s^2 \sigma_v^2 - \sigma_{sv}^2)$ is the determinant of $\mathbf{\Sigma}$. Since $\gamma > 0$, $|\Sigma| > 0$ and $\sigma_{sv} < 0$, requiring both $w_s^{mv} > 0$ and $w_v^{mv} > 0$ simultaneously results in the condition:

$$q_v > \max \left[\frac{\sigma_v^2}{\sigma_{sv}} q_s, \; \frac{\sigma_{sv}}{\sigma_s^2} q_s \right]. \tag{4}$$

We call the right-hand side of (4) the *optimal diversification threshold* for a long equity investor, based on the minimum-variance criterion. This is the expected return on volatility that would justify an investor with a long equity position to add a long position in volatility for the purpose of diversification. Notice that, since the covariance $\sigma_{sv} < 0$, this threshold is positive iff $q_s < 0$ and it does not depend on the investor's risk aversion γ.

Including additional risky assets such as bonds or commodities in the portfolio increases the dimension of the covariance matrix and makes the derivation of the explicit formula for the corresponding diversification threshold difficult. However, the general condition for all weights being simultaneously positive still does not depend on γ, because

$$\mathbf{w}^{\mathbf{mv}} = \gamma^{-\mathbf{1}}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{q} > \mathbf{0} \iff \boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{q} > \mathbf{0}. \tag{5}$$

3.2 Skewness Preference

Among others, Chunhachinda et al. (1997), Prakash et al. (2003) and Sun and Yan (2003) demonstrate that higher moments are highly relevant to the investor's portfolio decision. In particular, the incorporation of skewness is found to have a major impact on portfolio construction. This is particularly important for the problem in hand because Alexander et al. (2015), Guobuzaite and Martellini (2012) and others provide strong empirical evidence that returns on volatility futures are not normal, in particular because they have high positive skewness and excess kurtosis.

One of the assumptions underpinning MV optimization is that portfolio returns $\boldsymbol{\mu} \in \mathbb{R}^k$, are normally distributed. To be more general we could assume only that an investor has an exponential utility $U(\boldsymbol{\mu}) = -\exp(-\gamma\,\boldsymbol{\mu})$ where γ denotes the risk aversion. In this case the optimal portfolio weights are the same whether we maximize the expected utility $\mathbb{E}[U(\boldsymbol{\mu})]$ or its certainty equivalent $\mathrm{CE} = -\gamma^{-1}\log\left(\mathbb{E}[U(\boldsymbol{\mu})]\right)$. Indeed, when returns are normal the CE becomes the standard MV criterion.

When we extend our analysis to accommodate non-normal returns, still using the exponential utility assumption, the CE has no simple analytical form. Hence we compute the ex-ante allocations for optimal portfolios numerically. We take the empirical joint distribution of returns to derive portfolio returns for a given weights vector $\mathbf{w}$ and optimize these weights by maximizing the expected (exponential) utility of portfolio returns.

Our empirical study will incorporate an out-of-sample performance assessment, and for this we need a measure that converges to the ordinary SR as the skewness and excess kurtosis tend to zero. Just as the optimal allocation for a MV investor can be obtained by maximizing the SR, Hodges (1998) showed that the optimal allocation for a general investor is that which maximizes the generalized Sharpe ratio (GSR) defined as

$$\mathrm{GSR} = \left(-2\log(-\mathbb{E}^*[U(\boldsymbol{\mu})])\right)^{0.5}, \tag{6}$$

where $\mathbb{E}^*[U(\boldsymbol{\mu})]$ denotes the maximum expected utility obtained over all possible portfolio weights. The GSR is a simple transformation of the CE which indeed converges to the SR as returns become more normal. So that we can compare the GSR with the SR (which can be negative) by measuring performance for the skewness-aware investor with the standard approximation

$$\mathrm{GSR} \approx \mathrm{SR}\left(1 + \frac{skew}{3}\mathrm{SR}\right)^{0.5}. \tag{7}$$

However, we note that conclusions based on SR and GSR optimization should be regarded with caution, because errors in measurement of model parameters can lead to unstable allocations. This has been thoroughly discussed in the literature – see Chopra and Ziemba (1993) among others – and errors in the expected returns have much greater effect on results than errors in the covariance matrix forecast.

3.3 Personal Views and Equilibrium Returns

Black and Litterman (1992) argue that investors should not base their decisions entirely on historical data, or more generally on their own personal views about expected returns. Any long-term investment should also take account of equilibrium expected returns, and if investor's personal views are highly uncertain then the resulting allocations will be more stable than MV allocations, because they will not deviate too far from the equilibrium returns.

In this section we consider the volatility diversification problem from the perspective of an equity respectively equity-bonds investor who uses the classical Black-Litterman (BL) model, as interpreted and implemented by He and Litterman (1999). This assumes that asset returns follow a normal distribution with unknown expected returns vector $\boldsymbol{\mu} \in \mathbb{R}^k$, $\boldsymbol{\mu} = (\mu_1, ..., \mu_k)^T$ with $k \in \mathbb{N}$ the number of assets, and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{M}^{\mathbf{k \times k}}$. An expression for the posterior distribution of returns is obtained by conjugating two normal distributions, one for the investor's personal views and the other for the equilibrium returns.[1]

In a capital asset pricing model (CAPM) market equilibrium, all investors hold the market portfolio $\mathbf{w^M} \in \mathbb{R}^{\mathbf{k}}$ and share the same beliefs about expected returns, encapsulated by a normal prior distribution with mean $\boldsymbol{\mu}^M \in \mathbb{R}^k$ and covariance matrix $\zeta\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma} \in \mathbb{M}^{\mathbf{k \times k}}$ is the historical covariance matrix. The parameter $\zeta \in \mathbb{R}^+$ is a positive constant representing the uncertainty in the prior distribution for expected returns. Black and Litterman (1992) propose that ζ should be set close to zero, as the investor is more certain about the distribution of expected returns than for returns themselves. Since, under the i.i.d. assumption, the variance of a sample mean is inversely proportional to the sample size $n \in \mathbb{N}$, we follow He and Litterman (1999) and Blamont and Firoozy (2003) and set $\zeta = n^{-1}$. In addition to prior beliefs, which are based on equilibrium returns, an individual investor holds his own, subjective views about the distribution of expected returns. These views might be about the distribution of expected returns on individual assets, and/or about certain portfolios of these assets. The views are represented using a matrix $\mathbf{P} \in \mathbb{M}^{\mathbf{l \times k}}$, with l number of personal views, such that $\mathbf{P}\boldsymbol{\mu}$ follows a normal distribution with mean vector $\mathbf{q} \in \mathbb{R}^{\mathbf{l}}$ and diagonal covariance matrix $\boldsymbol{\Lambda} \in \mathbb{M}^{l \times l}$, which defines the investor's confidence in each view. Now, blending equilibrium with subjective views yields a posterior normal distribution for expected returns with mean $\boldsymbol{\mu}^{BL} \in \mathbb{R}^k$:

$$\boldsymbol{\mu}^{BL} = \left[(\zeta\boldsymbol{\Sigma})^{\mathbf{-1}} + \mathbf{P^T}\boldsymbol{\Lambda}^{\mathbf{-1}}\mathbf{P}\right]^{-1} \left[(\zeta\boldsymbol{\Sigma})^{\mathbf{-1}}\boldsymbol{\mu}^{\mathbf{M}} + \mathbf{P^T}\boldsymbol{\Lambda}^{\mathbf{-1}}\mathbf{q}\right] \tag{8}$$

and covariance matrix $\boldsymbol{\Theta} = \left[(\zeta\boldsymbol{\Sigma})^{\mathbf{-1}} + \mathbf{P^T}\boldsymbol{\Lambda}^{\mathbf{-1}}\mathbf{P}\right]^{-1}$. Then the assets' actual returns follow a normal distribution with the same mean, $\boldsymbol{\mu}^{BL}$, but the covariance matrix $\bar{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \boldsymbol{\Theta}$.

Finally, He and Litterman (1999) apply the MV optimizer to the posterior distribution for

[1] Several studies have extended the original Black-Litterman model allowing for other return distributions. See for instance, Martellini and Ziemann (2007), who take preferences about higher moments of asset return distributions into account or Giacometti et al. (2009) who apply t-student and the stable distributions and use alternative risk measures.

actual returns to obtain the solution for the unconstrained optimal portfolio weights:

$$\mathbf{w^{BL}} = (1+\zeta)^{-1}(\mathbf{w^M} + \mathbf{P}\boldsymbol{\lambda}), \tag{9}$$

where $\mathbf{w^M} = \gamma^{-\mathbf{1}}\mathbf{\Sigma^{-1}}\boldsymbol{\mu}^{\mathbf{M}}$ are the equilibrium portfolio weights and vector $\boldsymbol{\lambda} \in \mathbb{R}^l$ is given by:

$$\boldsymbol{\lambda} = \gamma^{-1}(1+\zeta)\mathbf{X^{-1}q} - \mathbf{X^{-1}P\Sigma w^M}, \tag{10}$$

with

$$\mathbf{X} = \mathbf{P\Sigma P^T} + \zeta^{-\mathbf{1}}(\mathbf{1}+\zeta)\mathbf{\Lambda}. \tag{11}$$

Black and Litterman (1992) and He and Litterman (1999) assume that $\mathbf{\Lambda} \in \mathbb{M}^{l\times l}$ is a diagonal matrix. Meucci (2005) relaxed this assumption, suggesting that $\mathbf{\Lambda}$ is directly proportional to $\mathbf{P\Sigma P^T}$. We follow Meucci (2005) and set

$$\mathbf{\Lambda} = \eta\mathbf{P\Sigma P^T}. \tag{12}$$

Thus the uncertainty in each personal view is proportional to the historical variance, with the same proportionality constant η.[2]

In our empirical work we consider two different portfolios: a portfolio with equity and volatility (PF1) and a portfolio with equity, bonds and volatility (PF2). The exposure to volatility is represented by futures and because these contracts are in zero net supply they have zero weights in the equilibrium portfolio. Hence, in our case the equilibrium weights are either $\mathbf{w}_1^M = (1,0)^T$ for PF1, or $\mathbf{w}_2^M = (w_s^M, w_b^M, 0)^T$ for PF2.

Now, in the BL model the equilibrium returns are obtained via reverse MV optimization,

[2]Note that a more restricting assumption, where η is set equal to ζ, has been used in the implementations of He and Litterman (1999) and Da Silva et al. (2009). We prefer to include η as a free parameter so that we can investigate how the BL solution behaves as the investor becomes relatively more or less confident in his own views, i.e. as η decreases or increases, respectively, but ζ remains fixed.

so that

$$\boldsymbol{\mu}^M = \gamma \mathbf{\Sigma} \mathbf{w}^{\mathbf{M}}. \tag{13}$$

The equilibrium expected returns vector will be denoted $\boldsymbol{\mu}_1^M = [\mu_s^M, \mu_v^M]^T$ or, when bonds are also included, we use $\boldsymbol{\mu}_2^M = [\mu_s^M, \mu_b^M, \mu_v^M]^T$ to denote the equilibrium returns for equities (μ_s^M), bonds (μ_b^M) and volatility (μ_v^M) assets.

Consider now two possibilities for the current views of the investor: (a) only one view on the volatility asset, i.e. $l = 1$, and (b) views on all assets in portfolio, i.e. $l = k$. In each case we now compute the minimum expected return on volatility that will justify a long position for it in the BL portfolio. That is, we generalize the threshold 4 which pertains to standard MV optimization to the case where an investor combines personal views with equilibrium expected returns:

Proposition 1. *Suppose the investor has only one view about expected returns on volatility. Then the optimal portfolio weights $\mathbf{w}^{\mathbf{BL}}$ in (9) are positive for all assets if, and only if, the excess return on the volatility asset, q_v, fulfills the following inequality:*

$$q_v > (1+\zeta)^{-1} \mu_v^M. \tag{14}$$

Proof. We prove (14) only for the portfolio with three assets as the proof is similar for the two-asset portfolio. For an investor with only one view on volatility, i.e. $l = 1$, the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{q}$ each have only one element, i.e. $\boldsymbol{\lambda} = \lambda_v$ and $\boldsymbol{q} = q_v$ and the matrix of views becomes $\mathbf{P} = [0, 0, 1]$. We also have

$$\mathbf{P\Sigma P^T} = \sigma_{\mathbf{v}}^{\mathbf{2}}, \quad \mathbf{P}\boldsymbol{\mu}^{\mathbf{M}} = \mu_{\mathbf{v}}^{\mathbf{M}}, \quad \mathbf{\Lambda} = \eta \sigma_{\mathbf{v}}^{\mathbf{2}}, \quad \mathbf{X} = \left(\mathbf{1} + \eta + \eta\zeta^{-\mathbf{1}}\right) \sigma_{\mathbf{v}}^{\mathbf{2}} > \mathbf{0}. \tag{15}$$

Using (9), the BL allocation becomes

$$\mathbf{w}^{BL} = (1+\zeta)^{-1} [w_s^M, w_b^M, \lambda_v]^T. \tag{16}$$

Consequently, $w_s^{BL} > 0$, $w_b^{BL} > 0$ for $w_s^M > 0$, $w_b^M > 0$. Requiring $w_v^{BL} > 0$ yields the condition: $\lambda_v > 0$, or equivalently $\gamma\lambda_v > 0$. Substituting (15) in (10) yields

$$\gamma\lambda_v = \left[(1+\zeta)q_v - \mu_v^M\right](1+\eta+\eta\zeta^{-1})^{-1}\sigma_v^{-2},$$

so $w_v^{BL} > 0$ if, and only if,

$$q_v > (1+\zeta)^{-1}\gamma\left[w_s^M\sigma_{sv} + w_b^M\sigma_{bv}\right] = (1+\zeta)^{-1}\mu_v^M, \tag{17}$$

where the last equality follows from (13). □

A fund manager with no personal views on any assets except for the volatility asset, will thus allocate positively to volatility whenever his expected return is greater than the equilibrium return, scaled for the uncertainty about the prior that is captured by the parameter ζ.[3]

We now consider the case where the investor has views on all assets (i.e. $l = k$) and derive the condition for a long position on volatility in the BL-optimal portfolio.

Proposition 2. *When an investor has asset-specific views on all assets with mean vector* $\mathbf{q}$*, then positive allocations to all assets* ($\mathbf{w^{BL}} > 0$) *are guaranteed if, and only if,*

$$\mathbf{\Sigma^{-1}q} > -\zeta^{-1}\eta\gamma\mathbf{w^M}. \tag{18}$$

For the two-asset portfolio (PF1) the condition (18) can be simplified to

$$q_v > \max\left[\frac{\sigma_{sv}}{\sigma_s^2}q_s, \frac{\sigma_v^2}{\sigma_{sv}}q_s + a\right], \text{ where} \tag{19}$$

$$a := \zeta^{-1}\eta\gamma\sigma_{sv}^{-1}\left|\Sigma\right|.$$

Proof. With (12) and $\mathbf{P} = \mathbf{I}$, the matrix $\mathbf{X}$ becomes $\mathbf{X} = x\mathbf{\Sigma}$, where $x = 1 + \zeta^{-1}(1+\zeta)\eta$. It

[3]However, it is independent of η, and would also be independent of ζ under the modification of the BL model suggested by Pézier (2007), which argues that, since $\mathbf{\Theta} = \zeta\mathbf{\Sigma}$ in the absence of any personal views, we should set $\boldsymbol{\mu}^M = \gamma(1+\zeta)\mathbf{\Sigma w^M}$ rather than $\boldsymbol{\mu}^M = \gamma\mathbf{\Sigma w^M}$, so that in (8) $\boldsymbol{\mu}^M$ should be replaced by $(1+\zeta)\boldsymbol{\mu}^M$ and (9) becomes simply $\mathbf{w^{BL}} = \mathbf{w^M} + \mathbf{P}\boldsymbol{\lambda}$. See Pézier (2007) for further details. The factor $(1+\zeta)^{-1}$ would not appear in (9) and consequently nor in (17), so that diversification would be optimal simply when the expected return on volatility exceeds its equilibrium return.

follows then

$$\mathbf{w}^{BL} > 0 \iff \gamma(1+\zeta)\mathbf{w^{BL}} = [\gamma\mathbf{w^M} + \gamma\boldsymbol{\lambda}] > \mathbf{0},$$

$$\iff \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M + (1+\zeta)x^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{q} - x^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^M > 0,$$

$$\iff \boldsymbol{\Sigma^{-1}}\mathbf{q} > (1-x)(1+\zeta)^{-1}\boldsymbol{\Sigma^{-1}}\boldsymbol{\mu}^{\mathbf{M}}.$$

The condition (18) for a positive allocation follows then with $(1-x) = -\zeta^{-1}\eta(1+\zeta)$ and $\boldsymbol{\Sigma^{-1}}\boldsymbol{\mu}^{\mathbf{M}} = \gamma\mathbf{w^M}$.

For the two-asset portfolio with $\mathbf{q} = (q_s, q_v)^T$ equation (18) yields

$$\boldsymbol{\Sigma^{-1}}\mathbf{q} = \frac{1}{|\Sigma|}\begin{pmatrix} \sigma_v^2 q_s - \sigma_{sv} q_v \\ \sigma_s^2 q_v - \sigma_{sv} q_s \end{pmatrix} > \begin{pmatrix} -\zeta^{-1}\eta\gamma \\ 0 \end{pmatrix}.$$

After some tedious algebra, we have two inequalities for q_v:

$$(I) \quad q_v > \sigma_{sv}^{-1}\left(\zeta^{-1}\eta\gamma\,|\Sigma| + \sigma_v^2 q_s\right),$$

$$(II) \quad q_v > \frac{\sigma_{sv}}{\sigma_s^2} q_s,$$

so the threshold for a positive allocation to volatility is the maximum of both terms. □

Remark 1. $a < 0$ since $\sigma_{sv} < 0$ and $|\Sigma| > 0$. Thus, the diversification threshold for the MV investor in (4) is greater than or equal to that for the BL investor in the two-asset portfolio. In other words, the diversification frequency for the MV investor will be no greater than that for the BL investor.

Remark 2. For $\eta = \zeta$ the condition (18) reduces to $\boldsymbol{\Sigma^{-1}}\mathbf{q} > -\gamma\mathbf{w^M}$. However, the condition (18) cannot be further simplified without any assumptions on the sign of the corresponding covariances, which is not always clear. While the correlation between equity and volatility is throughout negative, the correlation between bonds and equity or bonds and volatility fluctuates between positive and negative values. So for instance, the 1-year rolling correlation between the bonds fund AGG and the VIX ETN VXX ranged between -0.14 and 0.46 from January 2006 to June 2013.

4 Empirical results

4.1 Data

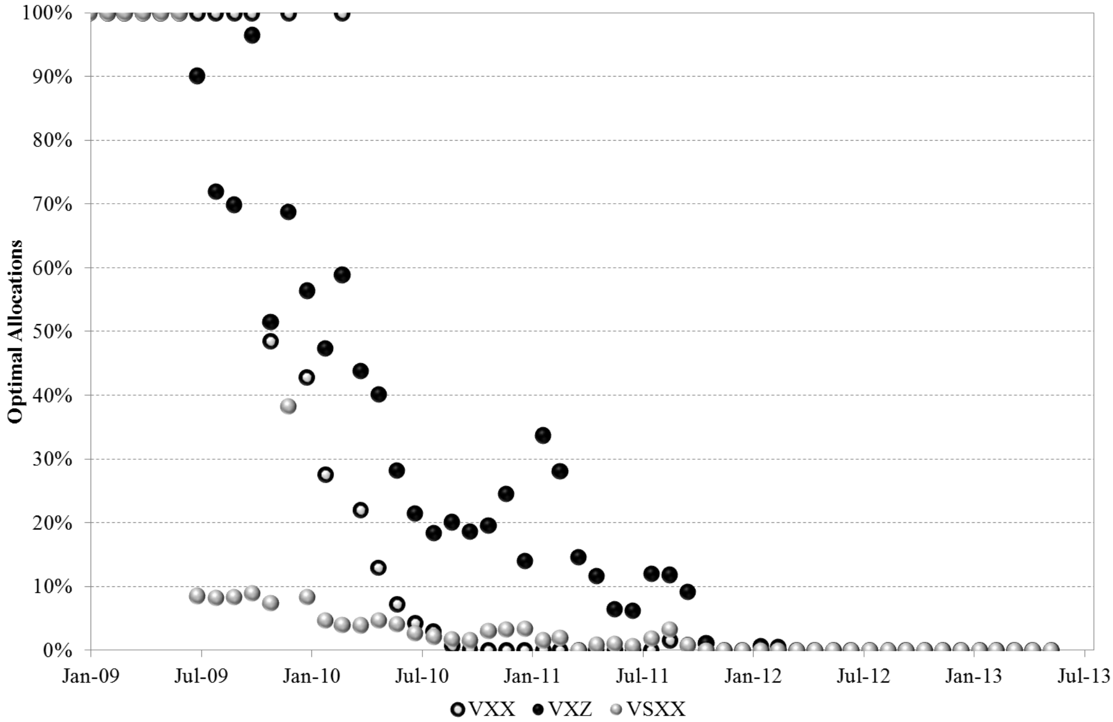
To assess the diversification impacts of volatility exposure we add volatility futures to two different portfolios: a pure equity portfolio and a traditional portfolio with stocks and bonds. We consider (a) a US investor who allocates between the S&P 500 ETF (SPY), iShares Barclays Aggregate Bond fund (AGG) and the short-term or mid-term VIX ETNs (VXX or VXZ), and (b) a European investor with positive allocations to the EURO STOXX 50 ETF (SX5EEX), iShares Barclays Euro Aggregate Bond fund (EUAGG) and VSTOXX short-term ETN VSXX.

From January 2006 to the end of June 2013, we employ 7.5 years of Bloomberg daily data on the closing, bid and ask prices of SPY, AGG, SX5EEX and EUAGG. In order to perform an in-sample and out-of-sample empirical analysis over this long data period we construct synthetic prices of constant-maturity portfolios of volatility futures with 1-month and 5-month maturities.[4] The prices of these daily-rebalanced portfolios are available as S&P constant maturity VIX futures and the VSTOXX short-term futures indices. They represent the indicative values for the ETNs that are now most highly traded, i.e. VXX (VIX 1-month volatility tracker), VXZ (VIX 1-month volatility tracker), and VSXX (European VSTOXX 1-month volatility tracker). Market data on these ETNs are only available since January 2009 for VXX and VXZ, respectively in April 2010 for VSXX.[5] Nevertheless, for convenience, in the following we denote our constant-maturity volatility futures series by these tickers. That is, when we refer to VXX, VXZ and VSXX we refer to the constant-maturity volatility futures portfolios which determine the indicative values of these ETNs. The 1-month US Treasury

[4]The alternative of rolling over the futures position at or soon before expiry has also been explored but made little quantitative and no qualitative difference to our results. In other words, the roll cost which dominates positions on constant-maturity volatility futures trackers can be taken daily, or periodically, without affecting results significantly.

[5]In fact the premiums are very small and switching to market prices as they become available, instead of using indicative values for the entire period, has a negligible affect on our results.

Figure 1: Ex-post Optimal Allocations to Volatility



1-month rolling optimal allocations to VXX, VXZ and VSXX, calculated using the MV criterion (1). Expected returns and covariances are based on the last three years of daily data.

bill as well as the 1-month EURIBOR rate are used for the risk-free rate, as published on Bloomberg.

4.2 Ex-Post Results

We consider an investor who allocates between equity, bonds and volatility rebalancing his portfolio monthly. Suppose that, at each rebalancing point, he uses the last three years of historical data to forecast returns and covariances and then chooses ex-post optimal allocations which maximize SR (or GSR, if he has skewness preference). Then Figure 1 displays the optimal allocations to volatility based on the mean-variance criterion (1), starting from January 2009.

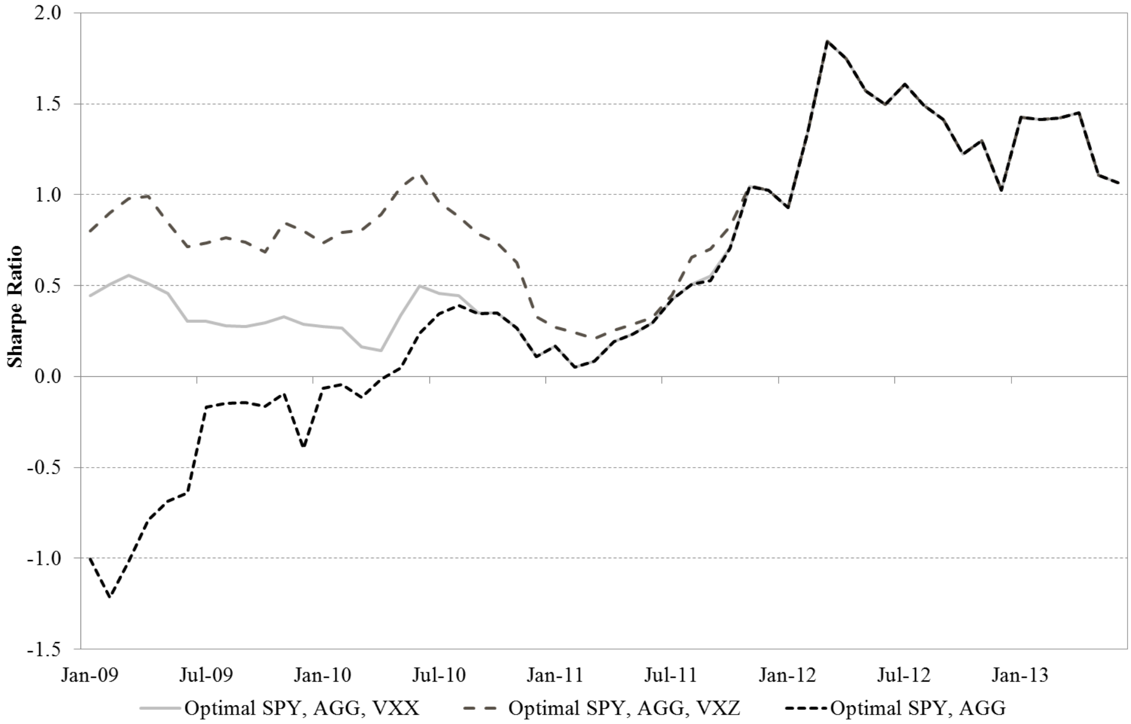
Given the three-year sample period used to forecast expected returns and covariances, the optimal allocations from 2009 to 2011 are based on historical data covering the financial crisis period when expected returns on equity and bonds were negative whereas expected returns on volatility were positive. Hence, from January to October 2009, the ex-post optimal portfolio diversified with VXX would consist of the VXX alone. Later on, the falling volatility derivatives market caused decreasing weights on VXX and a reallocation to bonds and equity markets. Since October 2010, the optimal portfolio has no exposure to VXX at all.

By contrast, the portfolio diversified with the mid-term volatility (VXZ) has more stable positive allocations to volatility and over a much longer period, which is consistent with previous findings of Hill (2013) and Whaley (2000). Since June 2009 the allocation to VXZ has decreased steadily but it remained a significant part of the portfolio until November 2011. For the European investor, however, it was ex-post optimal to invest in the VSTOXX short-term volatility futures during the first half of 2009 only. From July 2009 to October 2011 the optimal weights in VSXX remained under 10% and, like in the US portfolios, are zero from November 2011 onwards.

All volatility diversified portfolios have higher Sharpe Ratios (SRs) when compared to the portfolio without volatility exposure. Particularly during 2009, immediately after the financial crisis, including volatility in the portfolio would have generated a positive SR (and GSR) much greater than the SR (or GSR) of the optimal portfolio with equity and bonds alone. For instance, in May 2009, the SR of the ex-post optimal SPY/AGG portfolio was -0.69 compared to 0.46 of the portfolio diversified with VXX. Portfolios diversified with VXZ or European VSXX would achieve even higher SRs (0.85 resp. 0.73). Figure 2 compares the SRs for the portfolios diversified with VXX and VXZ with the SR of the equity-bonds portfolio(SPY, AGG) without volatility exposure. While the equity-bond portfolio generates negative SRs from January 2009 to April 2010, both volatility diversified portfolios have high positive excess returns. In particular, the SRs of the ex-post optimal portfolio diversified with the VXZ are double those of the VXX-diversified portfolio. However, the out-performance diminishes with decreasing allocations to volatility and from December 2011 on both portfolios no longer have any volatility exposure.

To represent a skewness-aware investor we maximize the approximate GSR given by (7). As expected, given its positive skewness, the ex-post empirical results reflect a slightly higher optimal allocation to volatility ($10-20\%$) but there are only a few months where the diversification to volatility would be ex-post optimal based on the GSR but not on the SR criterion.

Figure 2: Ex-post Analysis for Optimal Portfolios



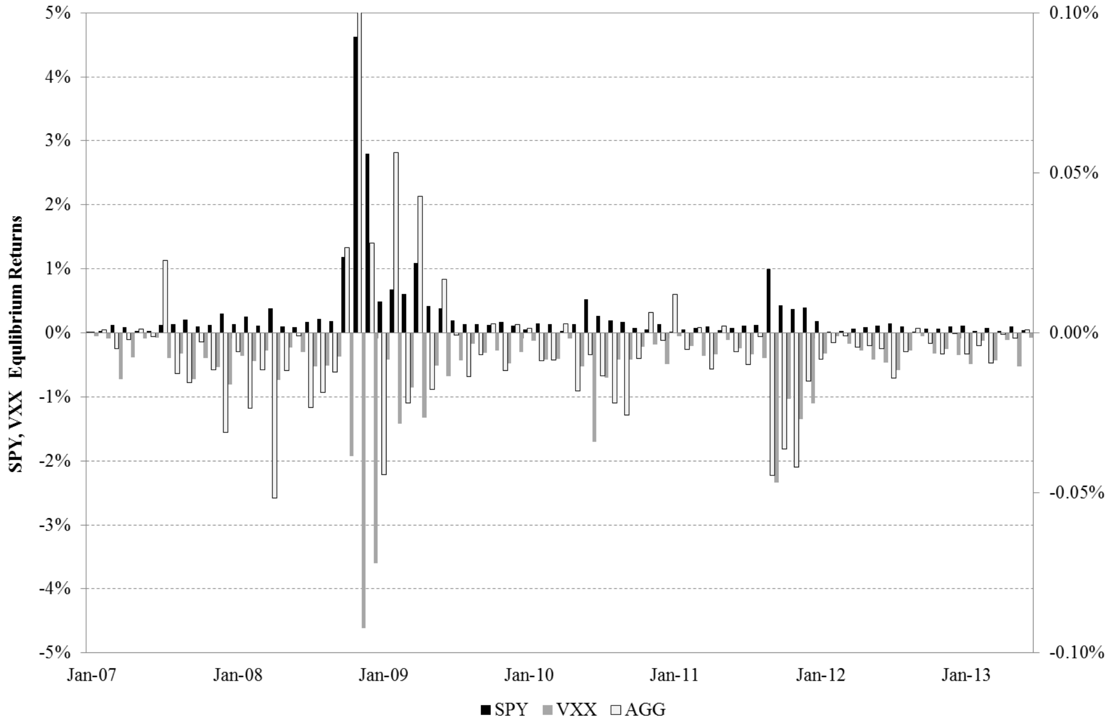
1-month rolling Sharpe Ratios for the ex-post optimal diversified portfolios with VXX (Optimal SPY, AGG, VXX) and VXZ (Optimal SPY, AGG, VXZ) and Sharpe Ratios of the optimal portfolio without volatility exposure (Optimal SPY, AGG). Expected returns and covariances are based on the last three years in-sample period. From December 2011 on, the three lines coincide because there is no allocation to volatility ETNs anymore.

The resulting risk/return ratios differ only marginally.[6] We conclude that adding volatility exposure to an equity-bond portfolio would have increased the overall portfolio return and decreased its standard deviation. However, it was ex-post beneficial to the mean-variance and/or skewness-aware investor to diversify into volatility only when the portfolio decision was based on data that included the credit and banking crises of 2007 and 2008.

As noted, these results are broadly in line with previous research based on ex-post analysis, although the length of in-sample period does vary between studies. However, what matters to investor seeking to implement our research is an ex-ante analysis, which we turn to in the next sub-section, and there the choice of in-sample period for computed expected returns and covariances will be varied and no longer be fixed at three years.

[6]For brevity, we report here only optimal allocations based on the SR criterion. Ex-post results for the optimal portfolios for the skewness-aware investor are available on request from the authors.

Figure 3: Equilibrium Returns for the US Portfolio



The equilibrium expected returns on equity (SPY), bonds(AGG) and volatility (VXX) between monthly rebalancing points. We assume that the investor has risk aversion $\gamma = 1$ and uses 1 month of daily data for estimations. The equilibrium market portfolio is set to $\mathbf{w^M} = (0.6, 0.4, 0)^T$.

4.3 Ex-Ante Optimal Diversification Frequency

The success of any ex-ante diversification strategy will depend on the quality of the forecasts for the returns, and the covariances around these expected returns. Focussing now on the Black-Litterman model, where personal views are combined with equilibrium returns, the historical mean vector $\mathbf{q}$ and covariance matrix $\mathbf{\Sigma}$ in the optimal portfolio (9) are based on daily data with an in-sample period of size n, covering either 1 month, 3 months or 12 months. This way we can generate a very long period for ex-ante results, starting in January 2007.[7] Following standard industry practice (e.g. as in Blamont and Firoozy (2003)) we set $\zeta = n^{-1}$ and consider two possible values for η, viz. $\eta = \zeta$ and $\eta = 1$. When $\eta = 1$ personal views are held with far greater uncertainty than equilibrium expected returns and optimal allocations deviate less from the equilibrium portfolio than when $\eta = \zeta$.

We compare three portfolios: The US short-term portfolio diversified with VXX, the US

[7]As noted later, using a longer in-sample period adds nothing of any qualitative value to our conclusions. However, these results are also available upon request.

mid-term portfolio diversified with VXZ and the European short-term portfolio diversified with VSXX. For each portfolio we consider three types of investors: a mean-variance investor (MV), a skewness-aware investor (SA) and a BL investor with asset-specific views on all assets (BL–3). For illustration, Figure 3 depicts the equilibrium returns used in the BL model for the three-asset US portfolio. These are calculated according to (13) with a historical covariance matrix based on the last month of daily data. The equilibrium market portfolio shown has weights vector corresponding to a widely used reference portfolio with 60% equity and 40% bonds and with a zero weight on volatility futures. We report the returns between each rebalancing point in monthly terms only for $\gamma = 1$, since equilibrium returns for other values of γ can easily be deduced from these, given (13). Equilibrium expected returns are positive for equity, negative for volatility, and they have a very large negative correlation. The equilibrium returns for bonds are very small and slightly negative ranging from $-0.5\% - -0.6\%$, except in November 2008 when US interest rates were cut sharply at the onset of the financial crisis resulting in a single outlier of 0.89%. The corresponding equilibrium returns for the European portfolio have similar size to US equilibrium returns and are available on request.

Starting in January 2007 at each monthly rebalancing point we suppose that each type of investor (i.e. MV, SA and BL) compares his expected returns and covariances (based on either 1 month, 3 months or 12 months of daily data) with the diversification conditions in (5) or (18). If the corresponding inequality is fulfilled, the investor holds the ex-ante optimal portfolio with a long position in volatility; otherwise, he holds equity only (or an equity-bond portfolio) until the next rebalancing point. We further assume that each investor has access to a 1-month risk-free asset for financing his investment.

Table 1 reports the proportion of days when diversification is perceived as optimal out a total of 78 monthly rebalancing points. Results are disaggregated according to the risk-aversion coefficient $\gamma = 1$ and $\gamma = 4$ and to the sample size n. For brevity, we present results based on daily data and with the equilibrium three-asset portfolio set to $\mathbf{w}^{\mathbf{M}} = (0.6, 0.4, 0)^T$, but the qualitative conclusions are similar when we use weekly data or different equilibrium weights. Note that when the investor has views on all assets ($\mathbf{P} = \mathbf{I}$) the optimal volatility weight does not depend on the equilibrium weights on equity or bonds since in this case

Table 1: Frequency of Optimal Volatility Diversification

	$\gamma = 1$			$\gamma = 4$		
n	1M	3M	12M	1M	3M	12M
US Short-Term Portfolio (SPY,AGG,VXX)						
MV	35%	35%	42%	35%	35%	42%
SA	36%	38%	44%	36%	38%	44%
BL3	35%	35%	42%	35%	35%	42%
US Mid-Term Portfolio (SPY,AGG,VXZ)						
MV	4%	8%	6%	4%	8%	6%
SA	3%	6%	6%	3%	6%	6%
BL3	4%	8%	6%	4%	10%	9%
EU Short-Term Portfolio (SX5EEX,EUAGG,VSXX)						
MV	32%	35%	32%	32%	35%	32%
SA	29%	35%	35%	29%	35%	35%
BL3	32%	35%	35%	32%	36%	41%

The proportion of rebalancing periods when diversification is perceived as optimal, i.e. the optimal weights to both equity and volatility assets are positive, respectively, for the mean-variance (MV), skewness-aware (SA) and Black-Litterman with views on all assets (BL–3) investors. For $(q_v, \mathbf{\Sigma})$ the corresponding investor uses the historical estimate based on 1 month, 3 months or 12 months of daily data. We set $\eta = \tau$, and $\mathbf{w^M} = (0.6, 0.4, 0)^T$.

$w_v^{BL} = x^{-1}\gamma^{-1}\mathbf{\Sigma^{-1}}q_v$.

When comparing different volatility products, it becomes apparent that all types of investors would choose to include the short-term volatility exposure in their portfolios much more frequently than the mid-term volatility exposure. Mid-term volatility (VXZ) is regarded as an optimal diversification instrument less than 10% of the time, whereas short-term volatility (VXX) is included in the optimal portfolio in 35%–42% of the rebalancing periods. A similar finding is apparent with the European short-term volatility (VSXX). Although slightly less frequent than the VXX, the VSXX is a thought to be significant part of the optimal portfolio during about 30% − −40% of the rebalancing periods.

As suggested by our theoretical results (see Remark 1 following Proposition 2), the MV investor diversifies less frequently than the BL investor. Nevertheless, the difference between the BL investor with views on all assets and a MV investor is quite small. Indeed, as already noted by Black and Litterman (1992), for $\eta \to 0$ the BL portfolio converges to the views portfolio when $\mathbf{P} = \mathbf{I}$. This can be easily seen in our setting as for $\eta \to 0$, the BL condition for the optimal diversification in (18) approaches the MV condition in (5). As $\eta \to \infty$ the BL (posterior) portfolio converges to the equilibrium (prior) portfolio as the views become less and less informative. Also the difference between the MV and SA diversification frequency is negligible. While the skewness-aware investor increases the diversification frequency to VXX by approximately $3-11\%$, there is no significant increase for the VSXX and even a slight decrease for the VXZ.

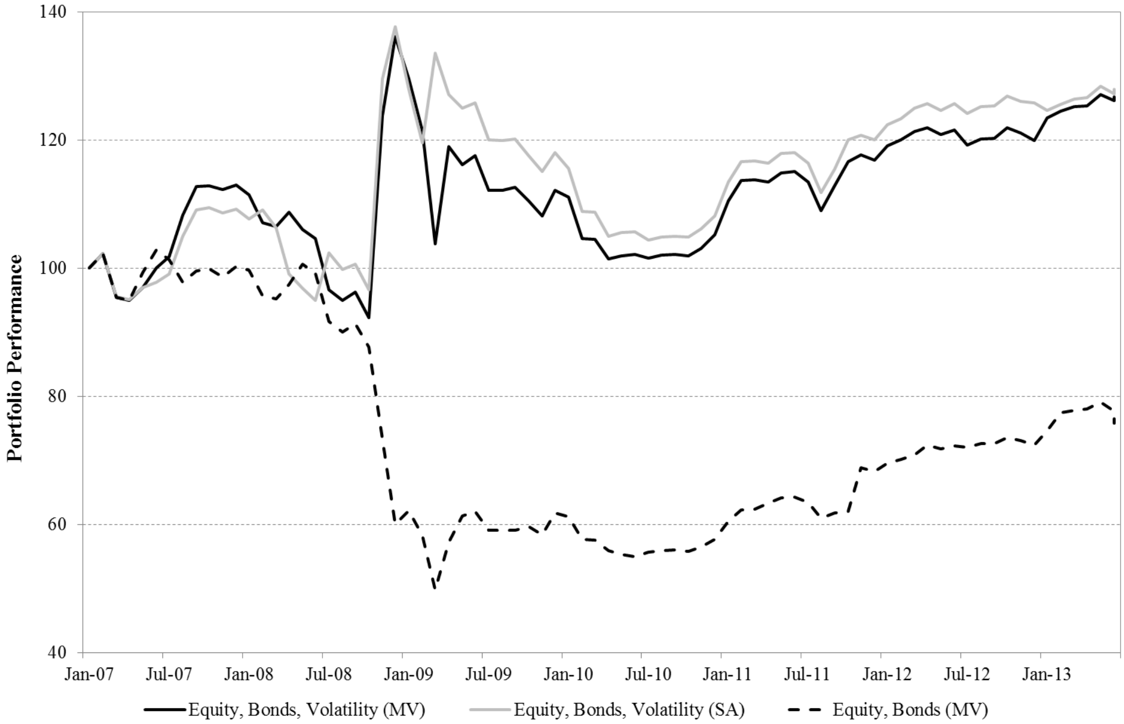
For brevity, only the results for the case $\eta = \zeta$ are reported in detail, this being the case that is most commonly assumed in the literature. Increasing η yields in a higher diversification frequency for the BL investor, since a negative view on volatility is held with less confidence and so the investor having views on correlations may still diversify, due to the typical negative correlation between equity and volatility. This effect is particularly apparent when diversifying using mid-term volatility, e.g. diversification becomes up to five times more frequent when $\eta = 1$. Results for the $\eta = 1$ case are available on request.

Diversification frequencies for the MV investor are identical for both $\gamma = 1$ and $\gamma = 4$ (as expected, since we have seen in (5) that the decision of the MV investor does not depend on the level of his risk aversion). In the BL model the diversification frequency should increase with γ as the right-hand side in equation (18) is decreasing in γ, hence diversification is more likely to occur as the investor becomes more risk-averse. However, empirically, we found the increase of γ from 1 to 4 to have only a very small effect. Much higher (perhaps unrealistic) levels of risk aversion would need to be assumed in order to affect the diversification frequency significantly.

We conclude that the maturity of the volatility futures is by far the most significant factor determining the investor's decision whether to diversify into volatility. Neither the optimization model, the risk tolerance nor the in-sample period used to determine the values for the model parameters influence this decision as much as the volatility maturity.

Table 1 clearly demonstrates that short-term volatility exposure is perceived as more suitable for equity diversification than the mid-term volatility, by all three types of investor. This observation contrasts with our ex-post analysis, which typically found a better performance from including mid-term rather than short-term volatility in an US equity-bond portfolio.

Figure 4: Comparison of Equity-Bonds and Volatility Diversified Portfolios



Portfolio growth of a theoretical \$100 investment in the optimal mean-variance (MV) equity-bonds portfolio (SPY/AGG) and in the ex-ante optimal portfolios diversified with the VIX short-term ETN VXX based on the MV and SA criterion starting in January 2007. The portfolio growth of the BL diversified portfolio is very close to that of the MV portfolio and is omitted for clarity. The optimization is based on $\gamma = 1$, $n = 21$ days, $\eta = \zeta$ and $\mathbf{w}^{\mathbf{M}} = (0.6, 0.4, 0)^T$.

Another potential contrast between the ex-post and ex-ante empirical analysis is that the perceived benefits from US short-term volatility diversification may become more frequent when using a longer sample period for the forecasts. Although the difference between the 1-month and 3-month results is quite small, the optimal diversification frequency increases from about 35% to about 42% when using the 12-month instead of the 1-month in-sample period. In the following we test whether the out-of-sample performance is indeed better when forecasts are based on 12 months of historical data, or whether the ex-ante optimally diversified portfolio actually do better when the investor used a shorter sample for making forecasts (and therefore chooses to diversify less frequently).

4.4 Out-of-Sample Performance Analysis

When analyzing the out-of-sample performance of the optimally-diversified portfolios we take care to include the bid-ask spread at each rebalancing point. It is important that performance is reported net of transactions costs because these are relatively high on volatility futures, as noted by Alexander et al. (2015) and others. We consider an investor who uses the previous month of historical daily data on SPY, AGG and VXX to make forecasts for expected returns $\mathbf{q}$ and covariances $\mathbf{\Sigma}$. Every month, from January 2007 till June 2013, he runs the optimization based either on MV, SA or BL criterion using $\mathbf{q}$ and $\mathbf{\Sigma}$ and compares the optimal weights with the previous month's portfolio. If a reallocation between equity, bonds and volatility is required, the additional cost in terms of corresponding spreads, times the change in the absolute weights, is subtracted from the portfolio return. The optimal portfolio is then held until the next rebalancing point when the optimization is repeated.

Figure 4 depicts the evolution of an investment of \$100 invested in: (a) the mean-variance optimal combination of SPY and the bonds funds AGG; (b) the mean-variance optimal combination of SPY, AGG and the short-term ETN VXX; and (c) the GSR optimal combination of SPY, AGG and VXX. The performance of the BL optimal portfolio diversified with VXX is very close to that of the MV portfolio and is hence omitted for clarity. Starting in January 2007, the equity-bonds portfolio without volatility exposure loses nearly 50% of its value at the onset of the financial crisis in 2008–2009. It gains in value from the middle of 2010 due to recovering stock markets but cannot compensate the dramatic loss of 2008 and closes in June 2013 with a total return of -24.2%. By contrast, both volatility diversified portfolios benefit from the negative correlation between SPY and VXX, especially during the financial crisis 2008–2009. Even after including the rebalancing costs in terms of spreads (which are 46 bps per month on average) volatility diversified portfolios generate much higher total returns. Over the period studied, they are at least 50% greater than the return on the equity-bonds portfolio.

Table 2 provides summary performance statistics for the MV optimal SPY/AGG portfolio and for the optimal volatility diversified portfolios based on MV, SA and BL criterion. Given an average return of -2.9% (annualized), the negative skew and high kurtosis, and a maximal monthly draw-down of 18.1%, the MV optimal SPY/AGG portfolio would scarcely have been an attractive investment during the last 6.5 years. By contrast, all volatility diversified portfolios performed quite well. Even though the portfolio volatility remains about the same

Table 2: Portfolio Performance

	Annualized Mean	Volatility	Skew	Kurtosis	Max. Monthly Draw-down	Sharpe Ratio	Total Return
Equity, Bonds (MV)	-2.9%	16.2%	-1.06	8.48	-18.1%	-0.24	-24.2%
Equity, Bonds, Volatility (MV)	5.2%	18.4%	3.28	24.48	-14.5%	0.23	26.7%
Equity, Bonds, Volatility (SA)	5.1%	17.2%	4.20	30.82	-7.1%	0.24	27.9%
Equity, Bonds, Volatility (BL)	5.1%	18.2%	3.24	24.25	-14.5%	0.23	26.7%

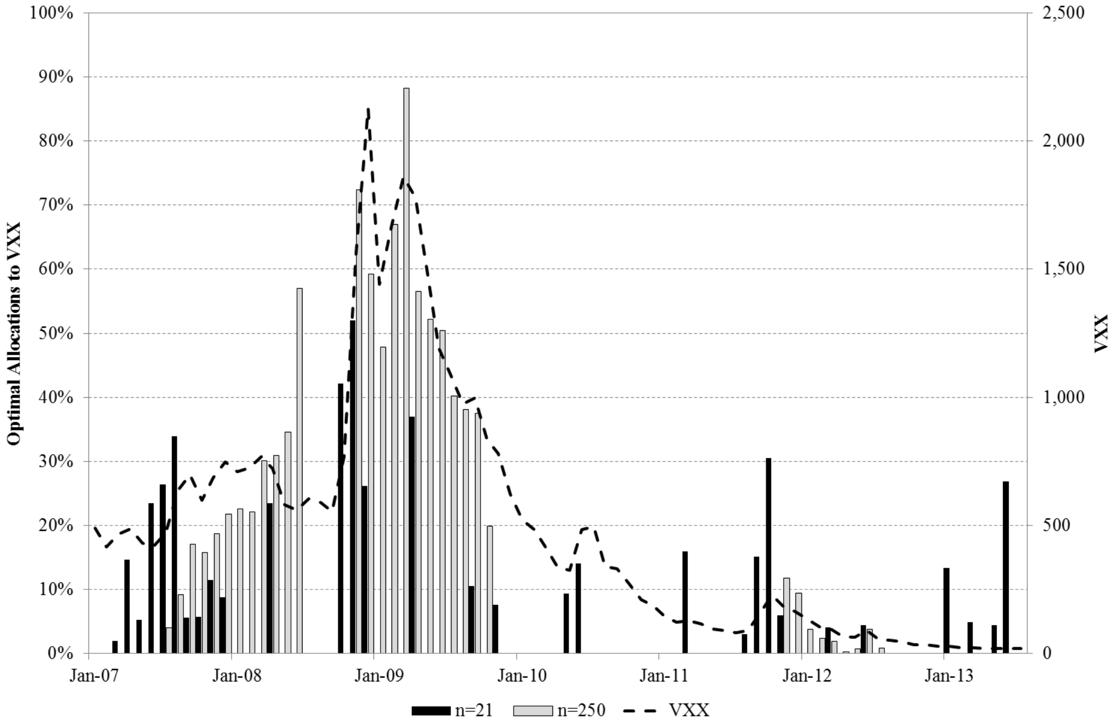
Performance of ex-ante optimal portfolios diversified with VXX based on the Mean-Variance, Skewness-Awareness and Black-Litterman optimization in comparison to the optimal MV portfolio without volatility exposure. The optimal portfolios are rebalanced monthly from January 3, 2007 to June 28, 2013 and forecasts for expected returns and covariance are based on the previous month of daily data. Rebalancing costs are included using actual bid-ask spreads on volatility futures. We use $\gamma = 1$ and $n = 21$, and set $\eta = \zeta$ and $\mathbf{w^M} = (0.6, 0.4, 0)^T$.

and kurtosis nearly triples, all portfolios have positive skewness. Each of them achieves an average annualized return of over 5% and total returns of over 26%. Including volatility in the portfolio also reduces the maximum draw-down, especially when applying the skewness-aware criterion to optimize the portfolio allocations.

However, one should be very careful when interpreting these results and concluding there are many benefits to diversification with volatility futures. The overall portfolio performance is highly sensitive to parameter inputs. For instance, the optimal MV volatility diversified portfolio would lose over 40% of its value with a negative average return of -8.2% p.a., when using the last 12 months ($n = 250$ trading days) instead of the last month ($n = 21$ trading days) for parameter estimations.

Figure 5 illustrates this issue using mean-variance optimality, and the (indicative) VXX as the volatility diversifier. While the optimal weights on VXX based on the times series of the more recent 21 days reflect almost exactly the price jumps in VXX, it takes much longer for the portfolio optimized using the last 250 observations to adjust the VXX exposure. For example, from October 2008 to November 2009, with a rolling window based on 21 days, it was optimal to diversify in VXX at only six of the monthly rebalancing points. However, it was perceived as optimal to diversify in VXX at nearly every rebalancing point when forecasts were based on 250-day rolling windows. This is because the great performance of short-term VIX futures during the banking crisis still influenced decisions one year later.

Figure 5: Optimal Allocations to VXX vs VXX Evolution



Optimal allocations to VXX based on MV criterion (2) using $n = 21$ and $n = 250$ days for estimation of returns and covariances.

5 Conclusions

Markets in volatility futures and their associated exchange-traded products are booming. In the U.S. alone, during 2013, nearly $3bn were traded daily on VIX futures, on average, and trading volume can exceed $10bn per day when we include VIX futures exchange-traded notes and funds.

Even though positions on volatility futures have lost value over the last few years, due to the unusually high roll cost associated with a market that is predominately in contango, their strong negative correlation with equity and high positive skewness could justify their inclusion as a diversifier to equity and bond portfolios.

Previous academic support for the benefits of VIX futures diversification in an ex-ante framework is meagre. Virtually all evidence in favor employs variance swaps or spot VIX, which have very different characteristics to VIX futures, or else the empirical study is based entirely on an ex-post analysis. Yet only an ex-ante analysis can be used in practice, and so the comprehensive ex-ante analysis and empirical study that we have presented is an essential contribution to the literature on this very important topic. Hardly any studies have modeled the decisions that would be made by asset managers that optimize, ex-ante, their holdings in risky assets using portfolio theory. Most papers perform only an empirical analysis that allocates to volatility using an ad-hoc rule, and all lack a thorough out-of-sample analysis based on ex-ante optimized portfolios. Moreover, much of the published work is confined to a relatively small sample which ends shortly after the banking crisis.

We have provided an extensive theoretical study on diversification of equity-bonds exposure first within a mean-variance optimization framework, later including skewness preference and then allowing for moderation of personal forecasts by equilibrium returns as used in the Black-Litterman framework. We have derived a general formula for the ex-ante "optimal diversification threshold", i.e. the expected return on a futures instrument that is sufficient to justify diversification by adding a long position in this contract to a portfolio which already contains any number of risky assets. This optimal diversification condition depends on the parameters of the optimization framework that is employed by the investor, including: the covariances between returns on equity, bonds and volatility; the expected returns on equity and bonds; and also on the investor's risk aversion.

Our empirical study employs data from 2006 to 2013, a longer period than any previous studies on the benefits of volatility diversification. We find that diversification of equity-bond portfolios by adding long positions in volatility futures would frequently have been perceived as optimal, ex-ante, by both US and European investors. However, the optimally-diversified portfolios out-performed traditional equity-bond portfolios only during crisis periods. In particular, for the European investor it was rarely ex-post beneficial to include volatility assets and for both investors only when the decision was based on the financial crisis data. Further, after accounting for transactions costs – including the roll costs on volatility futures which arise because the term structure is almost always in contango – we find that all the investors considered (i.e. standard mean-variance, skewness-aware investors and those holding personal views within the Black-Litterman framework) can benefit from volatility diversification, but only when forecasts are based on very recent market data. And in this case the same investors would only perceive that diversification into volatility is optimal during periods of market turmoil. Moreover, they would choose short-term futures as their preferred diversification instrument. These findings contradict the advice that is being given to investors by the exchanges that provide markets in these products.

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