Prices and Investment

with

Collateral and Default

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Abstract: This paper uses the framework of an OLG economy with three-period lived agents in which a durable good serves as collateral for loans, to study the effect of an unanticipated income shock when the economy is in a steady state equilibrium. We focus on the consequence of default on loans when the value of the collateral falls below the value of the debt it secures. We analyze the impulse response functions of the price and production of the durable good and show that there is an asymmetry between the response of the price and investment of the durable good to a positive and a negative income shock arising from default on the collateralized loans.

1 Introduction

The recent financial crisis originated in the mortgage market where a decrease in house prices triggered extensive default on mortgages. Motivated by this episode we study the consequence of a shock which unexpectedly changes the price of a durable good used to collateralize loans. As in the collateral model of Kiyotaki-Moore (KM) (1997) we adopt the method widely used in macroeconomics of studying the effect of an unanticipated income shock when the economy is in a steady state equilibrium. Our model differs from KM since we focus on consumer durables rather than the durable capital (or land) which is the focus of their study. The main conceptual difference with KM however is that we assume that as soon as the value of the collateral falls below the value of the loan the borrower defaults. That is, we adopt the approach of the general equilibrium literature initiated by Dubey-Geanakoplos-Zame (1995) and Geanakoplos-Zame (1997) where the seizure of the collateral is the only penalty for default.[1]

Thus in our model a negative unexpected shock leads to default, a feature not taken into account in KM. To study the consequence of default we contrast the behavior of the equilibrium variables, in particular the price and the production of a durable good, following a positive or negative income shock of the same magnitude. A positive shock to the agents' endowment—their labor income—leads to an increase in the price of the durable good so that the agents' wealth increases for two reasons: the first is the direct effect of the increase in the labor income, the second is the indirect wealth effect of the increased value of the durable purchased in the previous period. The indirect effect is akin the the "multiplier effect" exhibited by KM following a positive shock to the endowments of entrepreneurs.

If agents had to pay their loans in all circumstances then the equilibrium response to a negative shock would be the negative counterpart of the response to a positive shock. However with collateralized loans which permit default, the agents who borrowed against the full value of their durable last period (in our model the young who become middle aged) are not exposed to the negative indirect wealth effect: the decrease in price of the durable good following a negative shock implies that the value of the collateral is less than the value of the loan and the agents default. Thus aggregate demand decreases less than it would if default were not permitted. This has two consequences, one for the prices, the other for investment. The decrease in the durable good price is less than it would otherwise have been, and the the demand for the consumption good also decreases less than it otherwise would. Thus investment has to decrease more to reestablish equilibrium on the

[1] This assumption is relatively realistic for the current US mortgage market where a mortgage loan is de facto a non-recourse loan.

consumption good market. This fits with the financing behavior of the intermediaries who receive payments on the loans incurred in the previous period and use the proceeds to finance investment and new loans in the current period. With a negative shock the intermediaries make a loss on loans which default and are forced to reduce financing. If we compare the reaction functions of prices and investment to a negative shock and their reaction to a positive shock with a change in sign (the symmetric counterpart) then we find that the decrease in price is less steep while the decrease in investment is steeper.

We study these effects in the setting of an overlapping generations model (OLG) with three-period lived agents, identical cohorts and two goods, one perishable (also called the consumption good) and one durable which serves to collateralize loans. This provides a setting in which there are natural borrowers (the young agents) who purchase a durable good which serves at the same time for consumption and as collateral for their loans. Each agent has an initial endowment consisting solely of the perishable good, small in youth, larger in middle age and zero in retirement, the life-cycle profile originally studied by Samuelson (1958). The young borrow to finance their consumption of the durable good and we assume that the collateral constraint is binding: as we will see this implies that the young can not anticipate on their future income in middle age. The need to post collateral thus endogenously justifies the very plausible assumption that "Junior Can't Borrow" against future labor income in the well-known analysis of Constantinides-Donaldson-Mehra (2002). The durable good is transferred across periods with depreciation. To maintain or increase the stock of durable there is a technology with constant returns which transforms the consumption good into the durable good with a lag of one period. At each date there are spot markets for the perishable and the durable goods, and loans subject to collateral are issued by competitive intermediaries who make zero profit.

The perishable good serves as the unit of account so that an equilibrium consists of a sequence of durable good prices and interest rates such that markets clear. Studying paths which revert to a steady state equilibrium after a shock requires an analysis of the steady state equilibria and their stability properties. The economy always has a steady state in which the interest rate is zero (there is no population growth): following standard terminology for the OLG model, we refer to this steady state as the Golden Rule with Collateral (GRC). Its stability properties are however different from the Golden Rule (GR) of standard OLG models with perishable goods: with only perishable goods, when the old agents have zero endowment, the Golden Rule is unstable.[2] In

[2] This was the property that Samuelson (1958) considered a huge deception since the GR had all the nice properties except stability and hence would never be achieved in the long run (by markets): the steady state which is stable and hence is achieved in the long run, offers disadvantageous terms to the old generation in their retirement.

contrast in our model with a durable good which serves as a store of value, the GRC is stable (saddle-point stable when the local dynamics is of dimension greater than 1) as long as the durable good is desired for consumption and its depreciation rate is not excessive. Even though the old agents have no labor income, they become 'rich' by carrying over stocks of the durable good to retirement. We show that the economy thus behaves like a 'classical' rather than a 'Samuelson' economy in the terminology of Gale (1973), meaning that the GRC is stable and other steady states when they exist are unstable. This striking property of the model with a durable good implies that we can restrict the analysis of the impact of an unanticipated shock to a shock around the GRC.

2 Relation to the Literature

Equilibrium models with collateral have been developed over the last fifteen years and are the subject of active research. Collateral constraints were introduced almost simultaneously in the macro literature by Kiyotaki-Moore (1997) and in the general equilibrium literature by Dubey-Geanakoplos-Zame (1995) and Geanakoplos-Zame (1997). The ensuing GE literature on collateral has split into two branches: the one studies models in which durable goods serve as collateral for borrowing, the other branch initiated by Geanakoplos (2003) and Kubler-Schmedders (2003) studies models in which financial securities serve as collateral. From the modeling point of view the first is closer to our paper, but in terms of seeking a simplified structure to enable qualitative properties of equilibrium to be derived, the latter is closer to our concerns.

The models with durable goods progressively incorporated more realistic features of collateralized loans and their terms and focused on proving existence of equilibrium in these more general settings. Araujo-Orrillo-Pascoa (2000) and Araujo-Fajardo-Pascoa (2005) study two alternative approaches to endogenizing collateral, while Poblete-Cazenave-Torres-Martinez (2013) add bankruptcy with protected assets to the model. Araujo-Pascoa-Torres-Martinez (2002) and Pascoa-Seghir (2009) show how the collateral model can be extended to an open-ended future with infinite-lived agents; a model closer in spirit to ours since it draws on the overlapping generations structure is that of Seghir-Torres-Martinez (2008) which introduces realistic features such as random lifetimes and bequests. However since our goal is to derive properties of the equilibrium rather than establishing existence, we are led to study a much simpler OLG economy.

All the above models potentially have default in equilibrium, but it is difficult to know whether or not default actually occurs without explicitly calculating the equilibrium. Most papers which present calculated equilibria are those for which the collateral is a security (since the structure of the equilibria is typically simpler), but one interesting exception is the paper of Araujo-Kubler-

Schommer (2012) which presents calculated examples of two period equilibria with a durable good as collateral. The emphasis of their study is on understanding, through examples, how the collateral levels chosen in equilibrium depend on the initial resources and preferences of the agents, and on the welfare properties of equilibria.

In general it is difficult to derive properties of the model when a durable good serves as collateral, since the durable good has the dual role of providing utility services and serving as collateral, complicating its role in equilibrium. Thus most papers which study examples and derive properties of collateral equilibria showing which levels of collateral are endogenously chosen, use the model in which a security (long-lived if the model has more that two periods) serves as collateral, and examine how the prices of the security acting as collateral differ from the present value of its dividends, and how much collateral is needed to obtain Pareto optimal allocations. Models with a Lucas tree acting as collateral are presented in Kubler-Schmedders (2003) and Brumm et al. (2013) with incomplete markets, and in Chien-Lustig (2010) and Gottardi-Kubler (2013) with complete markets. In a series of papers Geanakoplos (2003, 2010) and Fostel-Geanakoplos (2009, 2012) study the prices of securities used as collateral in economies where risk neutral investors have different probability beliefs, showing the role of leverage in influencing the volatility of prices.

All the above papers differ in an important respect from our model: they are all exchange economies, while we are interested in the effect of default on prices and investment in a production economy. A more important difference is that all these models study rational expectations equilibria in which all futures shocks are perfectly anticipated. Their equilibria are thus closer to our steady state equilibrium with collateral constraints than to the path of the economy following an unanticipated shock which is the focus of our study. If in our economy the negative aggregate shock to agents' income had been anticipated, the collateral constraint would have been tighter, if not set to the level of the depreciated value of the durable good in the worst case scenario. In most of the models mentioned above there is no default in equilibrium because the collateral is endogenously set in such a way that its value always exceeds the value of the debt it collateralizes.

We see our paper as a complement to the rational expectations GE literature. To understand the normal functioning of an economy the rational expectations assumption is surely the right starting point. However there are episodes in which innovations occur which are better described by unanticipated rather than anticipated shocks. As documented by Gorton (2009, 2010) the subprime mortgages issued in the years preceding the 2007 crisis, were designed to be held by the borrowers for two or three years and to be refinanced after an appreciation of the house price.[3] In

[3]Subprime mortgages were adjustable rate mortgages for which the interest rate was relatively low and fixed for

the same period, home equity loans and refinanced mortgages permitted existing homeowners to borrow up to the full value of their houses. These loans were almost certain to default in the case of a decline in house prices, but negative shocks to prices were assumed to be regional and diversification through a complex process of securitization was assumed to reduce the risks. A significant economy wide decrease in house prices in the US, which had previously not been observed, was basically not anticipated. Thus the situation immediately preceding the 2007 crisis more closely resembled the equilibrium of a model in which no decrease in house prices was anticipated, and agents could borrow up to the full value of the depreciated durable, than to the equilibrium of a model with rational expectations in which collateral constraints explicitly take into account the possibility of negative aggregate shocks.

Our model shows that collateral serves to cushion the effect of the shocks for the borrowers, but magnifies the effect for the lenders and hence on investment which is qualitatively consistent with the recent experience on the housing market (see Figure 1). In view of the simplified nature of the model we do not claim to be able to match the magnitude of the effects which followed the financial crisis. In our model intermediaries are passive institutions which collect the reimbursements of past borrowers, lending the proceeds to current borrowers and making zero profit in the process. Nothing as dramatic as the near collapse of the financial system that we have witnessed occurs in our model. A richer framework will be needed to explain this part of the story.

The paper is organized as follows: Section 3 presents the model. Section 4 studies an economy with log preferences: this serves the useful role of permitting a complete analysis of the dynamics of the collateral equilibrium to be obtained, and allows the effect of an unanticipated shock on the current equilibrium variables to be analytically derived. Section 5 extends the analysis to an economy with more general preferences, giving a criterion for the local stability properties of an arbitrary steady state in terms of properties of the aggregate excess demand function. Of the possible profiles for a steady state, saddle-point stability and instability are shown to be the only likely cases. For the case of CES utilities, under natural conditions on aggregate excess demand, the GRC is the only saddle-point stable steady state: this implies that there is a unique perturbed equilibrium which reverts to the GRC after an unanticipated shock. This permits us to study the impulse response functions of the price and investment for the durable good which exhibit both the asymmetry between a positive and a negative shock, and a strong reaction (overshooting) effect on the date following the income shock.

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two or three years, and would be set to about 6% above LIBOR thereafter, prompting the need for refinancing.

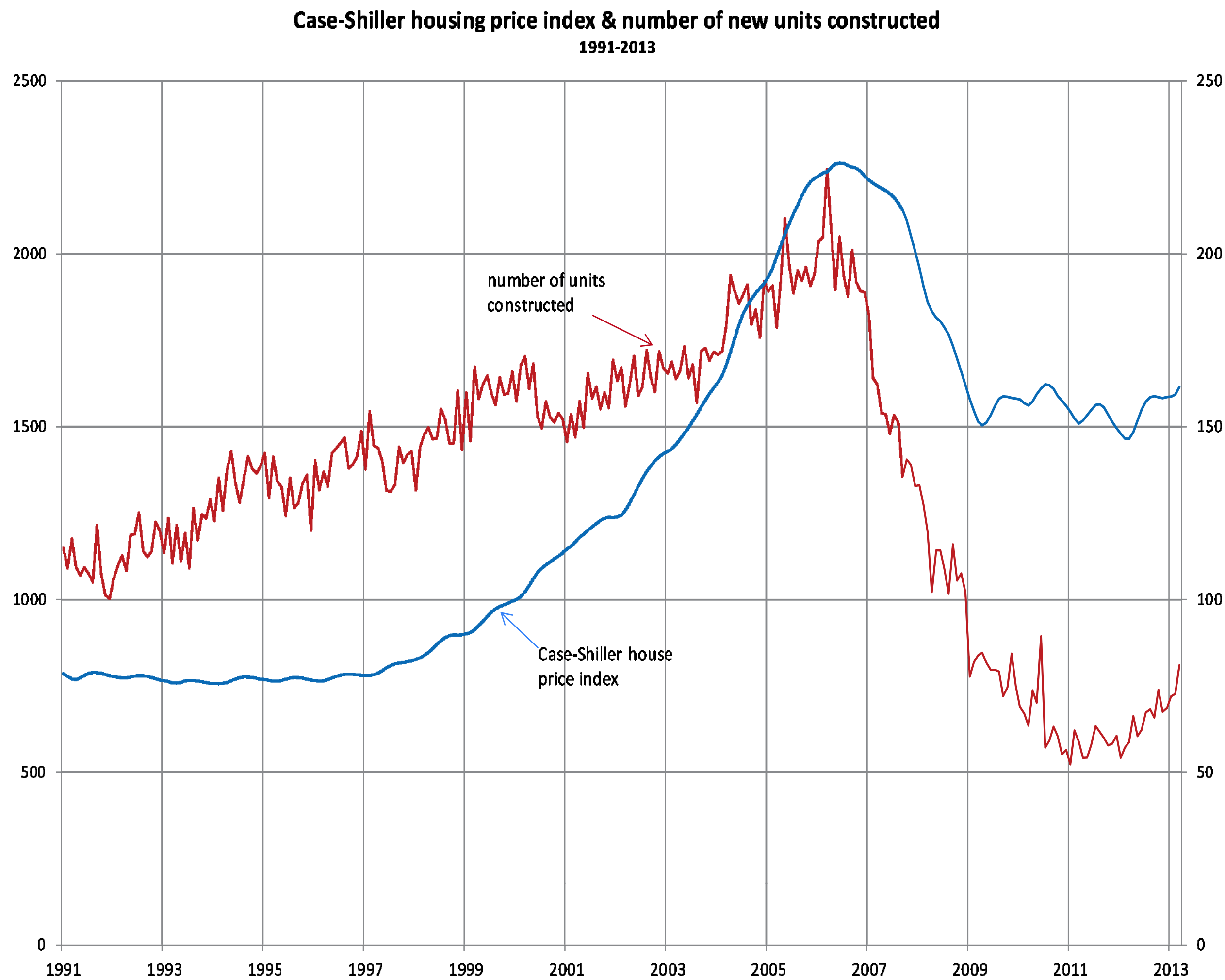

Figure 1: Figure 1 shows the number of New Privately Owned Housing Units Completed (in thousands, source: Department of Commerce) and the Case-Shiller Index of House Prices in 10 Large Cities in the US (source: Series CSXR, Standard & Poors-Dow-Jones) between 1991 and 2013. If we consider the introduction of the new forms of subprime mortgages and the related securitization in the late 90's as a positive 'innovation' or 'shock', it led to a very large increase of 125% in house prices and a smaller 40% increase in the number of new units built. The end of the rise in house prices in 2006 created a 'negative' shock which led to extensive default. Between 2006-2009, the price decrease was of the order of 33% while the decrease in the number of new units constructed was of the order of 65%.

3 Model

Consider an overlapping generations economy with two goods, one perishable and the other durable with depreciation rate $\delta > 0$ per period. Agents live for three periods as young, middle aged and retired: a new cohort of young agents of the same size enters at each date $t = 0, 1, \ldots$, while retired agents of the previous period exit. Thus at each date the three cohorts of young, middle and retired are of the same size. Every young agent enters with a lifetime endowment stream $(e^y, e^m, 0)$ consisting solely of the perishable good, and through trades on the markets obtains a lifetime consumption stream

$$(c, h) = (c^y, h^y, c^m, h^m, c^r, h^r)$$

where c and h denote the consumption of the perishable and durable good respectively, and the superscripts (y, m, r) refer to the stages of the agent's life. The preferences of every entering agent are represented by the same separable utility function U satisfying

Assumption $\mathcal{U}$: $U(c, h) = u(c^y, h^y) + \beta u(c^m, h^m) + \beta^2 u(c^r, h^r)$, where $0 < \beta \leq 1$ is the discount factor, and $u : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous, increasing, concave, and satisfies the Inada condition for both variables: $\lim u_c(c, h) \to \infty$ if $c \to 0$, for all $h > 0$, and $\lim u_h(c, h) \to \infty$ if $h \to 0$, for all $c > 0$.

Since the cohort sizes do not change we can focus on trades at each date between representative agents of each generation. At each date $t = 0, 1, \ldots$ there are markets for the two goods and a financial market for borrowing/lending. Let $(1, q_t)$ denote the spot prices of the two goods, the perishable good serving as the numeraire for the transactions of each period. Borrowing/lending takes place through competitive infinitely-lived intermediaries and we assume that there is no legal system for enforcing the payment of the debts, so that all borrowing must be guaranteed by collateral, an appropriate amount of the durable good which can be seized if the debt is not repaid.

While the stock of the perishable good $(e^y + e^m)$ available each period is exogenously given, the stock of durable good can be altered by production. The durable good is produced using the perishable good as input with a constant returns technology with a one-period lag, $y_{t+1} = \alpha z_t$, $\alpha > 0$, where z_t is the input of the perishable good invested at date t and y_{t+1} is the output of durable at date $t + 1$. If r_t denotes the interest rate on the market for loans between dates t and $t + 1$ then no-arbitrage between investing one unit of the perishable good in the loan market or in production implies implies

$$1 + r_t \geq \alpha q_{t+1} \tag{1}$$

z_t being equal to zero if the inequality is strict. We restrict attention to sequences of prices satisfying (1) for studying the optimal choice of the representative agent since otherwise the maximization of utility would not have a solution. Note that the units of the perishable good are determined by the choice of the values for (e^y, e^m) and the units of the durable good are determined by setting $\alpha = 1$: one unit of the durable is what can be produced with one unit of the perishable good.

Consider a young agent entering the economy at date t. His lifetime consumption $(c, h)_t = (c_t^y, h_t^y, c_{t+1}^m, h_{t+1}^m, c_{t+2}^r, h_{t+2}^r)$ will be obtained through purchases on the spot markets at dates t, $t+1$, $t+2$ at prices $(1, q_t), (1, q_{t+1}), (1, q_{t+2})$ when he is young, middle aged and retired. These purchases will be financed by income obtained from his lifetime endowment $(e^y, e^m, 0)$, from the sale of the depreciated previously purchased durable good $((1-\delta)h_t^y, (1-\delta)h_{t+1}^m)$, from borrowing (b_t^y, b_{t+1}^m) on the loan market, and from investment (z_t^y, z_{t+1}^m) in the production of the durable good. The consumption stream and portfolio $\left((c, h)_t, b_t^y, b_{t+1}^m, z_t^y, z_{t+1}^m\right)$ of an agent entering at date t must satisfy the sequence of budget equations

$$c_t^y + q_t h_t^y + z_t^y = e^y + b_t^y \tag{2}$$

$$\begin{aligned} c_{t+1}^m + q_{t+1} h_{t+1}^m + z_{t+1}^m &= e^m - \min\{b_t^y(1+r_t),\ q_{t+1}(h_t^y(1-\delta) + z_t^y)\} \\ &\quad + q_{t+1}(h_t^y(1-\delta) + z_t^y) + b_{t+1}^m \end{aligned} \tag{3}$$

$$c_{t+2}^r + q_{t+2} h_{t+2}^r = q_{t+2}\left(h_{t+1}^m(1-\delta) + z_{t+1}^m\right) - b_{t+1}^m(1+r_{t+1}) \tag{4}$$

In addition the agents' borrowing (b_t^y, b_{t+1}^m) from the intermediaries in youth and middle age must satisfy the collateral constraints

$$b_t^y(1+r_t) \leq q_{t+1}(h_t^y(1-\delta) + z_t^y) \tag{5}$$

$$b_{t+1}^m(1+r_{t+1}) \leq q_{t+2}(h_{t+1}^m(1-\delta) + z_{t+1}^m) \tag{6}$$

and investment must be non-negative $(z_t^y, z_{t+1}^m) \geq 0$.

In (5) and (6) we assume that agents can use both the durable good that they buy and that produced by their investment, as collateral for their loans. As in Kiyotaki-Moore (1997) we assume that agents can borrow up to the point where the reimbursement due next period is equal to the value of the collateral guaranteeing the loan. This is a natural borrowing constraint in a deterministic economy. Were the collateral constraint looser, there would be arbitrage opportunities; were it tighter, it would constrain borrowing more than necessary, and competition among the lenders would push the borrowing limit up to the full value of their collateral.[4]

[4] In a finance economy with infinite-lived agents Chien-Lustig (2010) and Gottardi-Kubler (2014) extend the analysis to stochastic economies with complete markets for collateral constraints which are such that there is no default in equilibrium.

We focus on endowment profiles for which the collateral constraint (5) binds for the young agents. The constraint (6) for the middle aged agents never binds: for if it were binding, in view of (4), the agent would have no income and hence no consumption when retired. Thus while (6) is included for consistency, it is omitted in the analysis that follows.

In the perfect foresight deterministic case, the 'min' in the budget equation (3) can be omitted since the collateral constraint (5) ensures that the debt is paid, in which case (3) can be replaced by the budget equation

$$c_{t+1}^m + q_{t+1}h_{t+1}^m + z_{t+1}^m = e^m - b_t^y(1+r_t) + q_{t+1}(h_t^y(1-\delta) + z_t^y) + b_{t+1}^m \tag{3'}$$

However when we consider an unanticipated income shock at some date, then the price of the durable differs from what was anticipated and the 'min' becomes relevant. In this case, at the date when the shock occurs, the middle-age budget constraint is given by (3).

Finally note that the agent does not inherit an endowment of the durable good when young or middle aged: this implies that the durable good purchased by the agent when retired is not bequested. That is, we assume that when retired agents exit, their durable good exits with them. This assumption simplifies the model and avoids the presence of a bequest motive in the maximum problem of an agent.

3.1 Simplified Maximum Problem.

The agent's maximum problem consists in choosing a consumption stream and portfolio $\Big((c,h)_t\Big)$ $\Big(b_t^y, b_{t+1}^m, z_t^y, z_{t+1}^m\Big)$ which maximizes utility $U((c,h)_t)$ subject to the budget equations (2)-(4) and the collateral constraint (5). We analyze the deterministic problem where constraint (3) in middle age is given by (3′). This problem has the interesting property that if the collateral constraint (5) is binding it decomposes into a simple one-period problem for the agent when young, independent of what happens later in life, and a two-period problem for the agent in middle age, choosing consumption for middle age and retirement. This arises from the fact that a collateral-constrained young agent can not borrow against future income. These properties can be seen from the first-order conditions which, in conjunction with the constraints (2)-(5) characterize the solution to the agent's maximum problem. Letting $(\lambda_t^y, \lambda_{t+1}^m, \lambda_{t+2}^r)$ denote the multipliers induced by (2), (3′), (4), and μ_t the multiplier for (5), the FOC are:

$$
\begin{aligned}
&\text{(young)} & c_t^y: &\quad u_c^y = \lambda_t^y & (7)\\
& & h_t^y: &\quad u_h^y = \lambda_t^y q_t - (\mu_t + \lambda_{t+1}^m) q_{t+1}(1-\delta) & (8)\\
& & b_t^y: &\quad \lambda_t^y = (\lambda_{t+1}^m + \mu_t)(1+r_t), \quad \mu_t(b_t^y(1+r_t) - q_{t+1}(h_t^y(1-\delta) + z_t^y)) = 0 & (9)\\
& & z_t^y: &\quad \lambda_t^y \geq (\lambda_{t+1}^m + \mu_t) q_{t+1}, \quad = \text{ if } z_t^y > 0 & (10)\\
&\text{(medium)} & c_{t+1}^m: &\quad \beta u_c^m = \lambda_{t+1}^m & (11)\\
& & h_{t+1}^m: &\quad \beta u_h^m = \lambda_{t+1}^m q_{t+1} - \lambda_{t+2}^r q_{t+2}(1-\delta) & (12)\\
& & b_{t+1}^m: &\quad \lambda_{t+1}^m = \lambda_{t+2}^r (1+r_{t+1}) & (13)\\
& & z_{t+1}^m: &\quad \lambda_{t+1}^m \geq \lambda_{t+2}^r q_{t+2}, \quad = \text{ if } z_{t+1}^m > 0 & (14)\\
&\text{(retired)} & c_{t+2}^r: &\quad \beta^2 u_c^r = \lambda_{t+2}^r & (15)\\
& & h_{t+2}^r: &\quad \beta^2 u_h^r = \lambda_{t+2}^r q_{t+2} & (16)
\end{aligned}
$$

where the partial derivatives are evaluated at the optimal decision, and expressions like $u_c(c_t^y, h_t^y)$ have been abbreviated to u_c^y indicating the period of life and the partial differentiation.

Proposition 1 (Decomposition of choice problem with binding collateral constraint)

(a) *If $(c,h)_t$ maximizes $U((\tilde{c},\tilde{h})_t)$ under the sequential budget constraints (2), (3′), (4) and the collateral constraint (5) binds ($\mu_t > 0$) then*

(i) *(c_t^y, h_t^y) maximizes $u(\tilde{c}_t^y, \tilde{h}_t^y)$ under the constraint*

$$\tilde{c}^y + \left(q_t - \frac{q_{t+1}(1-\delta)}{1+r_t}\right)\tilde{h}^y = e^y \tag{17}$$

(ii) *$(c_{t+1}^m, h_{t+1}^m, c_{t+2}^r, h_{t+2}^r)$ maximizes $u(\tilde{c}_{t+1}^m, \tilde{h}_{t+1}^m) + \beta u(\tilde{c}_{t+2}^r, \tilde{h}_{t+2}^r)$ under the present-value budget constraint*

$$\tilde{c}^m + \left(q_{t+1} - \frac{q_{t+2}(1-\delta)}{1+r_{t+1}}\right)\tilde{h}^m + \frac{1}{1+r_{t+1}}(\tilde{c}^r + q_{t+2}\tilde{h}^r) = e^m \tag{18}$$

(b) *Conversely if $(c,h)_t$ is such that (c_t^y, h_t^y) satisfies (i) and $(c_{t+1}^m, h_{t+1}^m, c_{t+2}^r, h_{t+2}^r)$ satisfies (ii) and if*

$$
\begin{aligned}
u_c(c_t^y, h_t^y) &> \beta(1+r_t) u_c(c_{t+1}^m, h_{t+1}^m) & (19)\\
1 + r_{t+1} &\geq q_{t+2} & (20)
\end{aligned}
$$

then there exist a portfolio $(b_t^y, b_{t+1}^m, z_t^y, z_{t+1}^m)$ such that $((c,h)_t, b_t^y, b_{t+1}^m, z_t^y, z_{t+1}^m)$ maximizes U under the constraints (2), (3′), (4) and (5).

Proof: (a) Substituting the first statement in (9) into (8) we see that the FOCs (7)-(9) imply that the FOCs of problem (i) hold. Since $\mu_t > 0$,

$$b_t^y = q_{t+1}(h_t^y(1-\delta) + z_t^y)/(1+r_t) \tag{21}$$

Substituting the value of b_t^y in (21) into the sequential budget constraint (2) gives $c_t^y + q_t h_t^y + z_t^y = q_{t+1}(h_t^y(1-\delta)+z_t^y)/(1+r_t)$. Since either $z_t^y = 0$ or $q_{t+1}/(1+r_t) = 1$, (17) holds. In the same way, when (13) is substituted into (12), it is clear that the FOCs (11)-(16) imply that the FOCs for the problem in (ii) are satisfied. To show that (18) holds, add the sequential budget constraint (3) to the present value of (4) at date $t+1$ i.e. (4) multiplied by $1/(1-r_{t+1})$. Since either $z_{t+1}^m = 0$ or $q_{t+2}/(1+r_{t+1}) = 1$, (18) holds.

(b) Let $(c,h)_t$ be a solution of the problems in (i) and (ii) with budget constraints (17) and (18). We need to show that $(c,h)_t$ is also solution of the original problem of maximizing $U((\tilde{c},\tilde{h})_t)$ subject to the sequential budget constraints (2),(3′),(4), and the collateral constraint (5). Let $\tilde{\lambda}_t$ denote the multiplier for (17) and $\tilde{\lambda}_{t+1}$ the multiplier for (18). Then $(c,h)_t$ satisfies the FOCs for the problems (i) and (ii)

$$u_c^y = \tilde{\lambda}_t, \qquad u_h^y = \tilde{\lambda}_t\Big(q_t - \frac{q_{t+1}(1-\delta)}{1+r_t}\Big) \tag{22}$$

$$u_c^m = \tilde{\lambda}_{t+1}, \qquad u_h^m = \tilde{\lambda}_{t+1}\Big(q_{t+1} - \frac{q_{t+2}(1-\delta)}{1+r_{t+1}}\Big) \tag{23}$$

$$u_c^r = \frac{\tilde{\lambda}_{t+1}}{1+r_{t+1}}, \qquad u_r^h = \frac{\tilde{\lambda}_{t+1}}{1+r_{t+1}} q_{t+2} \tag{24}$$

Let $(\lambda_t^y, \lambda_{t+1}^m, \lambda_{t+2}^r)$ and μ_t be defined by

$$\lambda_t^y = \tilde{\lambda}_t, \quad \lambda_{t+1}^m = \beta\tilde{\lambda}_{t+1}, \quad \lambda_{t+2}^r = \beta^2 \frac{\tilde{\lambda}_{t+1}}{1+r_{t+1}}, \quad \mu_t = \lambda_t^y - \lambda_{t+1}^m(1+r_t)$$

(19) implies that $\mu_t > 0$. Let $(b_t^y, b_{t+1}^m, z_t^y, z_{t+1}^m)$ satisfy

$$b_t^y - z_t^y = c_t^y + q_t h_t^y - e^y \tag{25}$$

$$b_{t+1}^m - z_{t+1}^m = c_{t+1}^m + q_{t+1} h_{t+1}^m - e^m \tag{26}$$

with $z_t^y \geq 0, = 0$ if $q_{t+1} < 1 + r_t$, $z_{t+1}^m \geq 0, = 0$ if $q_{t+2} < 1 + r_{t+1}$. (25) implies that the sequential budget constraint (2) is satisfied, it also implies

$$b_t^y - z_t^y = \frac{q_{t+1}(1-\delta)h_t^y}{1+r_t} \iff b_t^y(1+r_t) = q_{t+1}(1-\delta)h_t^y + z_t^y(1+r_t)$$

and since either $z_t^y = 0$ or $1 + r_t = q_{t+1}$, the collateral constraint (5) holds with equality. In view of (21), (26) implies that (3) holds: (3′) combined with (18) implies

$$\frac{q_{t+1}(1-\delta)}{1+r_{t+1}} h_{t+1}^m + \frac{(c_{t+2}^r + q_{t+2} h_{t+2}^r)}{1+r_{t+1}} = b_{t+1}^m - z_{t+1}^m$$

Since either $z_{t+1}^m = 0$ or $1 + r_{t+1} = q_{t+2}$, (4) holds. It is straightforward to check that the FOCs (7)-(16) are satisfied, so that $(c, h)_t$ maximizes $U((c, h)_t)$ under the sequential budget constraints (2), (3′), (4) and the collateral constraint (5), which completes the proof. □

Remark. Normally adding a constraint complicates a choice problem. Proposition 1 shows that the opposite is true when we add the collateral constraint. For the agent's lifetime consumption choice problem is simplified when the collateral constraint binds: the constraint decomposes the lifetime problem into two simpler problems, one for consumption in youth, the other for consumption in middle age and retirement. Since a young agent who is borrowing constrained can not satisfactorily solve the trade-off between consumption when young and consumption in the later stages of life, the agent simply spends as much it he can given his current income e^y and the down payment that must be made to acquire the durable good. In view of (17) the choice between the consumption good and the durable is straightforward since the price the agent pays to obtain one unit of the durable good is the down payment $q_t - \frac{q_{t+1}(1-\delta)}{1+r_t}$ which depends on its resale value, its depreciation, and the interest rate on the loan.

When the agent gets to middle age, he simply uses the resale value of the durable brought over from youth $((1-\delta)h_t^y + z_t^y)$ to pay for his debt. With the debt disposed of, the agent starts afresh with a new two-period problem over middle age and retirement with future income stream $(e^m, 0)$. The resale value of the durable acquired in middle age reduces its effective price to the downpayment $q_{t+1} - \frac{q_{t+2}(1-\delta)}{1+r_{t+1}}$, while the durable purchased in retirement costs q_{t+2} since there is no following period in which to sell it. Since in middle age the borrowing constraint does not bind, the sequential budget constraints in middle age and retirement can be reduced to a singe present-value constraint in period $t + 1$.

Note that the investments z_t^y and z_{t+1}^m do not appear in the budget constraints (17) and (18) and, when $1 + r_\tau = q_{\tau+1}$, $\tau = t, t+1$, they are not determined by the optimal choices of the agents. This is clear from the two equations (25) and (26) which only determine the differences $b_\tau^i - z_\tau^i$, $\tau = t, t+1$ for $i = y, m$. The indeterminacy comes from the fact that investment is a constant returns activity, and that any agent who invests z units of the consumption good in the production of the durable can automatically borrow the full value z (which serves as collateral) the value $q_{t+1}z$ of the production next period guaranteeing the reimbursement $(1 + r_t)z$ of the debt. Investing in the durable and borrowing to cover the cost is a zero profit, constant returns activity that the young and middle-aged agents can undertake on any scale in every period. The aggregate level of investment $z_t = z_t^y + z_t^m$ will be determined by the equilibrium conditions, but the distribution between the investment undertaken by the young z_t^y and that undertaken by the middle-aged z_t^m is indeterminate.

3.2 Collateral Equilibrium.

Since we are interested in studying equilibria which include the steady state equilibria, we define an equilibrium over $\mathbb{Z}$, the set of all negative and positive integers.

Definition 1. An *equilibrium* is a sequence $\Big((c,h)_t, z_t, q_t, r_t\Big)_{t\in\mathbb{Z}}$ such that for each $t \in \mathbb{Z}$

(i) $(c,h)_t$ maximizes $U((c,h)_t)$ subject to (2)-(5)

(ii) $c_t^y + c_t^m + c_t^r + z_t = e^y + e^m$

(iii) $h_t^y + h_t^m + h_t^r = z_{t-1} + (1-\delta)(h_{t-1}^y + h_{t-1}^m)$

If for some date $t \in \mathbb{Z}$ the collateral constraint (5) for the young binds, we say that it is a *collateral equilibrium*. An equilibrium is called a *steady state* equilibrium if all the variables are constant in time

$$\Big((c,h)_t, z_t, q_t, r_t\Big) = \Big((c,h), z, q, r\Big), \qquad \forall\, t \in \mathbb{Z}$$

Note that in a steady state equilibrium, investment is positive $z = z^y + z^m > 0$, since the depreciated durable good must be replaced at each date: thus in a steady state equilibrium $q = 1 + r$ must hold.

Definition 2. If the pair $\Big((c,h), q, r\Big)$ satisfies

(a) $(c^y, h^y) \in \text{argmax}\Big\{u(\tilde{c}^y, \tilde{h}^y) \;\Big|\; \tilde{c}^y + \Big(q - (1-\delta)\Big)\tilde{h}^y = e^y\Big\}$

(b) $(c^m, h^m, c^r, h^r) \in \text{argmax}\Big\{u(\tilde{c}^m, \tilde{h}^m) + \beta u(\tilde{c}^r, \tilde{h}^r) \;\Big|\; \tilde{c}^m + (q - (1-\delta))\,\tilde{h}^m + \frac{1}{1+r}(\tilde{c}^r + q\tilde{h}^r) = e^m\Big\}$

(c) $u_c^y\,(c^y, h^y) > \beta\, u_c^m\,(c^m, h^m)$ (d) $(q, r) = (1, 0)$

then it is called the *Golden Rule Collateral steady state* (GRC).[5]

[5] We use the terminology "Golden Rule" because of the property that $r = 0$, i.e. the interest rate is equal to the rate of growth of the population, which is a characteristic property of the Golden Rule equilibrium in OLG economies without imperfections.

Proposition 2 (GRC steady state).

(i) *For any* $e^m > 0$*, there exists* $\underline{e^y} > 0$ *such that for* $0 < e^y < \underline{e^y}$*, the economy with endowments* $(e^y, e^m, 0)$ *has a GRC steady state.*

(ii) *A GRC steady state is a collateral equilibrium.*

Proof: (i) It is clear that there exists a solution (c, h) of the maximum problems in (a) and (b) when the prices are $(q, r) = (1, 0)$, since the two budget sets are compact and u is continuous. Since for the solution (c, h) of (a) and (b) we must have $h^y > 0$ it follows from (17) that $c^y < e^y$. If $e^y \to 0$ then $c^y \to 0$ and by the Inada condition $\frac{u_c(c^y,h^y)}{u_c(c^m,h^m)} \to \infty$. By continuity there exists $\underline{e^y}$ such that for $e^y \leq \underline{e^y}$, the binding collateral condition (c) is satisfied. Thus a GRC steady state of the economy with endowments $(e^y, e^m, 0)$ with $0 < e^y \leq \underline{e^y}$ exists.

(ii) Since by Proposition 1 the agent's lifetime choices are optimal given $(q, r) = (1, 0)$, it only remains to show that (a), (b) and (d) of Definition 1 imply that the market clearing conditions hold. Adding (17) and (18) gives

$$c^y + c^m + c^r + \Big(q - (1-\delta)\Big)(h^y + h^m) + qh^r = e^y + e^m$$

and, since $q = 1$, $c^y + c^m + c^r + h^y + h^m + h^r - (1-\delta)(h^y + h^m) = e^y + e^m$. Setting the output z of the durable so that

$$z = z^y + z^m = h^y + h^m + h^r - (1-\delta)(h^y + h^m)$$

ensures that the market clearing conditions for both the perishable good and the durable good are satisfied at all times, so that the GRC steady state is a collateral equilibrium. □

Condition (c) in Definition 2 ensures that the collateral constraint (5) of a young agent binds at the steady state. (i) in Proposition 2 shows that this condition can always be assured by suitably scaling income in youth relative to income in middle age: it thus amounts to a restriction on an agent's lifetime income stream $(e^y, e^m, 0)$.

We are interested in equilibria close to the GRC steady state equilibrium in which the collateral constraint binds and there is positive investment to replace the depreciated durable good. In this case the market-clearing equations for the consumption and durable good lead to the equilibrium difference equation

$$c_t^y + c_t^m + c_t^r + h_{t+1}^y + h_{t+1}^m + h_{t+1}^r - (1-\delta)(h_t^y + h_t^m) = e^y + e^m, \ t \in \mathbb{Z} \tag{27}$$

and the prices satisfy the relation $1+r_t = q_{t+1}$. The agent's demand functions implied by Proposition 1 can then be expressed as functions of the durable good prices $(q_t)_{t\in\mathbb{Z}}$, with (c_t^y, h_t^y) depending on q_t, while (c_t^m, h_t^m) depends on (q_t, q_{t+1}) and (c_t^r, h_t^r) on (q_{t-1}, q_t). Since the indices at date $t+1$ move up 1, the difference equation (27) defines the relation between the prices $(q_{t-1}, q_t, q_{t+1}, q_{t+2})$ at four dates that must be satisfied by an equilibrium. Provided we can solve this equation as $q_{t+2} = f(q_{t+1}, q_t, q_{t-1})$, a collateral equilibrium is described by a third-order difference equation.

4 Collateral Equilibrium with Log Utility

The difference equation (27) becomes simpler when agents in the economy have log preferences

$$u(c,h) = \ln(c) + \gamma \ln(h), \quad \gamma > 0 \tag{28}$$

Since agents spend a fixed share of their income on each good, the dependence of the demand on q_{t-1} and q_{t+2} in (27) disappears, and the difference equation reduces to a first-order difference equation. This makes it possible to do a complete analysis of the dynamics of a collateral equilibrium and to explicitly evaluate the change in the price and investment of the durable good induced by an unanticipated endowment shock at date 0.

4.1 Steady States and Dynamics

When the utility function $u(c,h)$ in Assumption $\mathcal{U}$ is given by (28), the parameters which characterize the economy are the two preference parameters (β,γ), where γ measures the relative desirability of housing, the depreciation rate δ of the durable good, and the agents' endowments of the consumption good (e^y, e^m) in youth and middle age. These endowments must be restricted to satisfy

$$e^y < \frac{e^m}{\beta(1+\beta)} \tag{29}$$

This condition implies that when $(c,h)_t$ solves the maximum problems with single budget constraints (i) and (ii) of Proposition 1, then the inequality (19) holds, ensuring that the collateral constraint is binding. The demands of the young, medium and retired deduced from Proposition 1 are given by

$$\begin{aligned} c_t^y &= \frac{e^y}{1+\gamma}, & h_t^y &= \frac{\gamma}{1+\gamma}\,\frac{e^y}{q_t-(1-\delta)} \\ c_t^m &= \frac{e^m}{(1+\gamma)(1+\beta)}, & h_t^m &= \frac{\gamma}{(1+\gamma)(1+\beta)}\,\frac{e^m}{q_t-(1-\delta)} \\ c_t^r &= \frac{\beta e^m q_t}{(1+\gamma)(1+\beta)}, & h_t^r &= \frac{\beta\gamma e^m}{(1+\gamma)(1+\beta)} \end{aligned} \tag{30}$$

It is clear that (29) implies that $u_c(c_t^y, h_t^y) = 1/c_t^y > \beta u_c(c_{t+1}^m, h_{t+1}^m) = \beta/c_{t+1}^m$.

Proposition 3. (Stability of GRC). *If agents' preferences in Assumption $\mathcal{U}$ are given by (28), and their endowments satisfy (29), then the equilibrium difference equation (27) is a first-order equation. Generically the economy has two steady state collateral equilibria, one of which is the Golden Rule with Collateral (GRC). Furthermore*

(i) *if* $\dfrac{\gamma}{\delta} > \dfrac{\beta e^m}{(1+\beta)e^y + e^m}$ *the GRC is stable, and the other steady state is unstable;*

(ii) *if* $\dfrac{\gamma}{\delta} < \dfrac{\beta e^m}{(1+\beta)e^y + e^m}$ *the GRC is unstable, and the other steady state is stable.*

Proof: With log preferences the equilibrium difference equation (27) is given by

$$\frac{e^y + \frac{e^m}{1+\beta}}{1+\gamma} + \frac{\beta e^m q_t}{(1+\gamma)(1+\beta)} + \frac{\gamma}{1+\gamma}\left(\frac{e^y + \frac{e^m}{1+\beta}}{q_{t+1} - (1-\delta)} + \frac{\beta e^m}{1+\beta} - (1-\delta)\frac{e^y + \frac{e^m}{1+\beta}}{q_t - (1-\delta)}\right) = e^y + e^m$$

which after the change of variable

$$x_t = q_t - (1-\delta)$$

can be written as

$$A x_t + \frac{B}{x_{t+1}} - \frac{(1-\delta)B}{x_t} - (B + \delta A) = 0$$

with

$$A = \frac{\beta e^m}{1+\beta}, \qquad B = \gamma\left(e^y + \frac{e^m}{1+\beta}\right).$$

Multiplying by $x_t x_{t+1}$ (which introduces the fictitious solution $x_t = x_{t+1} = 0$), the difference equation becomes

$$x_{t+1} = \frac{B x_t}{-A x_t^2 + (B + \delta A)x_t + (1-\delta)B} \tag{31}$$

The steady state solutions are the solutions of the equation

$$-Ax^2 + (B + \delta A)x - \delta B = 0 \tag{32}$$

which has two positive roots

$$x_1^* = \delta, \qquad x_2^* = \frac{B}{A} = \frac{\gamma}{\beta}\,\frac{(1+\beta)e^y + e^m}{e^m}$$

The first root corresponds to the Golden Rule with collateral (GRC) since $q = 1$, $r = 0$, implies $x = \delta$. To study which steady state is stable we write the difference equation (31) as $x_{t+1} = f(x_t)$. Then $f'(x) = \frac{B}{D^2}(Ax^2 + (1-\delta)B)$, where D is the denominator in (31). Thus $f'(0) = \frac{1}{1-\delta} > 1$ and $f'(\delta) = (1-\delta) + \frac{\delta^2}{x_2}$. $f'(\delta) < 1 \Longleftrightarrow x_2 > \delta$ which is equivalent to the condition in (i).

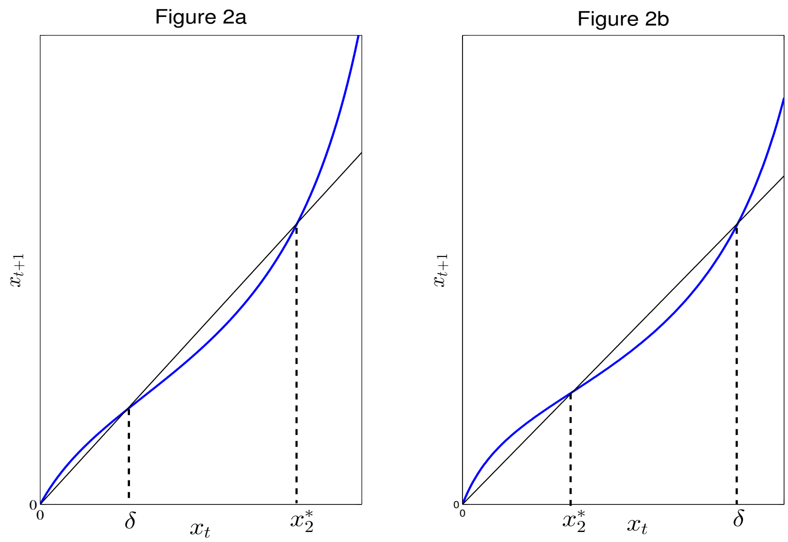

Figure 2: Golden Rule stable in Figure 2(a), unstable in Figure 2(b).

There are thus two possible cases which can arise: if (i) is satisfied the graph of f intersects the diagonal at $(0, \delta, x_2^*)$ in this order as shown in Figure 2(a), and if the reverse inequality in (ii) holds then the graph of f intersects the diagonal at $(0, x_2^*, \delta)$ (Figure 2(b)), from which the stability properties in (i), (ii) follow.[6] □

Remark. With log preferences, agents' demand functions have the Gross Substitute property and, despite the fact that the durable good is produced, our economy has the properties that have been established for multigood exchange economies with the Gross Substitute property:[7] namely that there are exactly two steady states, the Golden Rule (sometimes called the 'nominal' steady state since it requires the presence of an 'intermediary' or of 'money' to be feasible), and the 'no-intermediary' or 'real' steady state corresponding to the price $q_2 = (1-\delta) + x_2^*$, for which it can be shown that $b^y + b^m = 0$ so that borrowing and lending can be achieved by middle-aged agents lending to the young and being reimbursed in their retirement without the need for an intermediary.

Condition (i), which makes the Golden Rule (here the GRC) stable, requires that that the durable good be sufficiently desirable and durable (the depreciation rate δ sufficiently small). This is the case on which we focus attention since we are interested in economies where the durable good

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[6]We omit the non-generic case where $\frac{\gamma}{\delta} = \frac{\beta e^m}{(1+\beta)e^y + e^m}$ and there is a bifurcation at which the two steady states coincide.

[7]See Kehoe-Levine-Mas-Colell-Woodford (1991).

plays an important role—for example when the durable good is housing. At first sight it might seem that the economy falls into the 'Samuelson' category of Gale (1973) since agents have no endowment in retirement. However in our economy retired agents get income from their purchase of the durable good and from their investment in production in middle age. If (i) holds, then retired agents are sufficiently rich from bringing over the durable good into their retirement for the economy to fall into the 'classical' category where there is net aggregate borrowing ($b^y + b^m > 0$) by the young and middle-aged at the GRC, and the GRC is stable.[8]

4.2 Effect of an Unanticipated Income Shock at Date 0

Assuming that (i) of Proposition 3 holds, so that the GRC is the stable steady state, we examine the effect of an unanticipated shock at date 0 when the economy is at the GRC. The GRC equilibrium is such that

$$\begin{aligned}
&c^y = \frac{e^y}{1+\gamma}, && h^y = \frac{\gamma c^y}{\delta}, && b^y - z^y = \frac{\gamma(1-\delta)e^y}{\delta(1+\gamma)} \\
&c^m = \frac{e^m}{(1+\gamma)(1+\beta)}, && h^m = \frac{\gamma c^m}{\delta}, && b^m - z_m = \frac{e^m}{1+\beta}\left(\frac{\gamma}{1+\gamma}\,\frac{1-\delta}{\delta} - \beta\right) \\
&c^r = \frac{\beta e^m}{(1+\gamma)(1+\beta)}, && h^r = \gamma c^r && \\
&b^y + b^m = \frac{1}{1+\gamma}\left(\left(e^y + \frac{e^m}{1+\beta}\right)\frac{\gamma}{\delta} - \beta\frac{e^m}{1+\beta}\right), && && z \equiv z^y + z^m = \frac{\gamma}{1+\gamma}(e^y + e^m)
\end{aligned} \tag{33}$$

Suppose that there is a once and for all unanticipated shock to the agents' endowments at date 0: the endowments become

$$(e^y, e^m) \longrightarrow ((1+\Delta)e^y, (1+\Delta)e^m)) \equiv (e_0^y, e_0^m)$$

at $t = 0$ and revert to their previous values thereafter. Thus

$$(e_t^y, e_t^m) = (e^y, e^m), \qquad t \neq 0$$

$\Delta = \Delta e/e$ is the proportional change in the agents' endowments (and in the total endowment $e = e^y + e^m$). As we shall see the effect on the equilibrium is different depending on whether $\Delta < 0$ or $\Delta > 0$, which we write as Δ^- and Δ^+ respectively. Let Δq_0^- and Δq_0^+ denote the first-order

[8] The property that *positive net aggregate borrowing by the young and middle-aged cohorts is equivalent to the stability condition (i)* in Proposition 3 can seen from equation (33). It is interesting that this condition, proposed by Gale (1973) as the stability condition in the perishable good setting, is also the stability condition for our model with the durable good.

change in the price of the durable at date 0 following a negative or a positive shock and Δz_0^- and Δz_0^+ the change in investment. The analysis that follows shows the following properties

Proposition 4 (Date 0 response of prices and investment)

(i) $\Delta q_0^- < 0,\ \Delta z_0^- < 0, \quad \Delta q_0^+ > 0,\ \Delta z_0^+ > 0$

(ii) $\Delta q_0^- > -\Delta q_0^+, \quad \Delta z_0^- < -\Delta z_0^+$.

The price and investment of the durable good covary positively with the shock, but there is an asymmetry in the response to a positive and a negative shock. The price of the durable responds less (in absolute value) to a negative than to a positive shock, while the investment responds more. The reason is that the middle aged and the retired agents benefit from a capital gain for $\Delta > 0$, but the corresponding capital loss does not affect the middle aged when $\Delta < 0$ since they default on their loans. Let us show this formally:

Proof: We split the proof into the response to the negative and positive shocks respectively. Since the economy was following the steady state prior to date 0 when the shock occurs, the supply of houses at date 0 is in all cases

$$H_0 = (1-\delta)(h^y + h^m) + z \tag{34}$$

Negative Shock. We begin with the case Δ^- and assume that $q_0 < 1$ so that middle-aged agents default on their loans: as we will see this assumption is satisfied in the equilibrium we calculate. The demand of the young and the middle-aged has the same form as in (30), the incomes $(1+\Delta^-)e^y$ and $(1+\Delta^-)e^m$ replacing e^y and e^m. The middle-aged 'foreclose' rather than paying their debts and thus 'start fresh' at date 0. The demand of the retired does not have the same form as in (30) since their income is different from what they anticipated: the amount of durable good

$$\eta = (1-\delta)h^m + z^m$$

which they inherit from their middle age sells for a lower price than anticipated. Since the interest charged at date -1 is zero, their income is

$$I_0^r = q_0\eta - b^m \tag{35}$$

where we assume that the shock Δ^- is such that q_0, although less than 1, is sufficiently large for I_0^r to be positive, in which case the retired agents do not default on their loan b^m. Their demand for the durable good is then $h_0^r = \dfrac{\gamma I_0^r}{(1+\gamma)q_0}$ and the equilibrium equation on the durable good market becomes

$$\frac{\gamma}{1+\gamma}\left[\frac{(1+\Delta^-)(e^y+\frac{e^m}{1+\beta})}{q_0-(1-\delta)}+\eta-\frac{b^m}{q_0}\right]=H_0$$

Let $(\Delta q_0)^- = q_0 - 1$ denote the deviation of the durable good price at date 0 from the steady-state value induced by the shock in the agents' endowments.[9] Assuming that Δ^- is sufficiently small for a first-order approximation to be justified, the relation between $(\Delta q_0)^-$ and Δ^- is obtained by replacing $\dfrac{1}{q_0-(1-\delta)}$ by $\dfrac{1}{\delta}\Big(1-\dfrac{(\Delta q_0)^-}{\delta}\Big)$ and $\dfrac{1}{q_0}=\dfrac{1}{1+(\Delta q_0)^-}$ by $(1-(\Delta q_0)^-)$. This leads to

$$\left[-\left(\frac{1}{\delta}\right)^2\left(e^y+\frac{e^m}{1+\beta}\right)+b^m\right](\Delta q_0)^-+\frac{1}{\delta}\left(e^y+\frac{e^m}{1+\beta}\right)\Delta^-=0 \tag{36}$$

or

$$(\Delta q_0)^-=\frac{\delta\Delta^-}{1-\left(\dfrac{\delta^2 b^m}{e^y+\frac{e^m}{1+\beta}}\right)} \tag{37}$$

Since $b^m - z^m$ is constant, b^m is maximum when $z^y = 0$ and $z^m = z$, in which case

$$b^m=\frac{1}{1+\gamma}\left(\left(\delta e^y+\frac{e^m}{1+\beta}\right)\frac{\gamma}{\delta}-\beta\frac{e^m}{1+\beta}\right)<e^y+\frac{e^m}{\delta(1+\beta)}.$$

Thus the denominator is positive and $(\Delta q_0)^- < 0$: a decrease in the economy's endowment leads to a fall in the price of the durable at date 0, the price elasticity being of the order of, but larger than, δ since the denominator in (37) is less than 1.

The date 0 investment z_0 can be deduced from the date 0 market clearing condition for the perishable good

$$c_0^y+c_0^m+c_0^r+z_0=(1+\Delta^-)(e^y+e^m) \tag{38}$$

where the demand c_0^y and c_0^m have the form in (30) and $c_0^r=\frac{I_0^r}{1+\gamma}$. Let $(\Delta z_0)^- = z_0 - z$ denote the deviation of the date 0 investment from the steady-state value z in (33). Using first-order approximations for the changes leads to

$$(\Delta z_0)^-=n\Delta^- - m(\Delta q_0)^- \tag{39}$$

where

$$n=\left(1-\frac{1}{1+\gamma}\right)e^y+\left(1-\frac{1}{(1+\beta)(1+\gamma)}\right)e^m,\qquad m=\frac{\eta}{1+\gamma}. \tag{40}$$

[9]Since $q = 1$ at the GRC, Δq_0 is also the percentage deviation from the steady state price.

n is is the net contribution to the input for investment of the young and the middle-aged in the steady state, namely the difference between their endowment and their demand for the consumption good. Since $n > 0$ the first term is negative: the contribution of the young and the middle-aged to investment input decreases with the decrease in their endowment. The term $m(\Delta q_0)^-$ comes from the demand of the retired agents. A fall in q, which has mainly an income effect for the retired, leads to a fall in their demand for the consumption good, which increases the amount of the good available for investment. The first of the two terms in (39) dominates so that a negative shock reduces investment at date 0.[10]

Positive Shock. Now suppose there is a positive shock to the economy at date 0, so that agents' endowments are increased by the percentage Δ^+. We assume that the price q_0 which establishes itself on the durable good market is greater than 1 and this assumption will be justified below. Since the price of the durable exceeds its steady-state value, the middle-aged agents make a capital gain on the durable good inherited from their youth. Their income now comes from two sources: $q_0(h^y(1-\delta)+z^y) - b^y = (q_0-1)b^y$ from the sale of houses inherited from youth, and $e^m(1+\Delta^+)$ from the exogenous endowment. The equation for equilibrium on the durable good market which determines the price q_0 thus becomes

$$\frac{\gamma}{1+\gamma}\left[\frac{(1+\Delta^+)(e^y+\frac{e^m}{1+\beta})+\frac{(q_0-1)b^y}{1+\beta}}{q_0-(1-\delta)}+\eta-\frac{b^m}{q_0}\right]=H_0$$

where H_0 is the supply of durable good inherited from the previous period given by (34). Using the same approximations as in the derivation of (36) gives

$$\left[-\left(\frac{1}{\delta}\right)\left(e^y+\frac{e^m}{1+\beta}\right)+\frac{b^y}{1+\beta}+\delta b^m\right](\Delta q_0)^+ + \left(e^y+\frac{e^m}{1+\beta}\right)\Delta^+ = 0 \tag{41}$$

or

$$(\Delta q_0)^+ = \frac{\delta\Delta^+}{1-\dfrac{\delta^2 b^m}{e^y+\frac{e^m}{1+\beta}}-\dfrac{\frac{\delta b^y}{1+\beta}}{e^y+\frac{e^m}{1+\beta}}} \tag{42}$$

In view of the expression for b^m and b^y in (33) the denominator is positive: the positive shock Δ^+ leads to an increase in the price of durable good at date 0, the price elasticity being of the order of δ. However since the denominator in (42) is smaller than in (37), the volatility of the price is

[10]This can be seen by using the approximation $(\Delta q_0)^- \simeq \delta\Delta^-$ and taking the maximum value of η (when $z^m = z$). This leads to

$$(\Delta z_0)^- \simeq \frac{1}{1+\gamma}\left(\gamma\Big(1-\frac{\delta}{1+\gamma}\Big)e^y+\Big(\beta(1+\gamma)+\frac{\gamma(\gamma-\delta\beta)}{1+\gamma}\Big)\frac{e^m}{1+\beta}\right)(\Delta q_0)^-$$

which has the sign of $(\Delta q_0)^-$.

greater with a positive shock Δ^+ than with a negative shock Δ^-. The difference comes from the asymmetric way in which the unanticipated capital gain (loss) affects the income of the middle-aged. With a positive shock the middle-aged agents get a boost to their income coming from the induced capital gain, while the capital loss from a negative shock is passed to the intermediaries through default, but is not felt in the income of the middle-aged agents.

The effect of a positive shock to investment can be deduced from the market clearing on the consumption good market

$$\frac{(1+\Delta^+)(e^y+\frac{e^m}{1+\beta})}{1+\gamma}+\frac{q_0-1)b^y}{(1+\beta)(1+\gamma)}+\frac{q_0\eta-b^m}{1+\gamma}+z_0^m=(1+\Delta^+)(e^y+e^m)$$

If $(\Delta z_0)^+$ denotes the deviation investment from the steady state, this market clearing equation implies that

$$(\Delta z_0)^+ = n\,\Delta^+ - m^+\,(\Delta q_0)^+ \tag{43}$$

where n is given by (40) and $m^+ = m + \dfrac{b^y}{(1+\beta)(1+\gamma)} > m$ since $b^y > 0$. Comparing the price changes in (37) and (42) implies

$$\frac{(\Delta q_0)^+}{\Delta^+} > \frac{(\Delta q_0)^-}{\Delta^-}$$

and it follows that

$$\frac{(\Delta z_0)^+}{\Delta^+} = n - m^+\frac{(\Delta q_0)^+}{\Delta^+} < n - m\frac{(\Delta q_0)^-}{\Delta^-} = \frac{(\Delta z_0)^-}{\Delta^-}$$

□

The log utility case is valuable because it leads to explicit closed-form solutions for the equilibrium path of prices and investment following a shock to agents' endowments at date 0. The formulas (37), (39), (42) and (43) show explicitly the asymmetry between the (first-order approximations of) the equilibria following a positive or a negative shock. Two effects are present in each case: a direct effect on the income of the young and middle-aged agents, and an indirect effect from the change in value of the inherited stock of durable good. While the capital gain or loss affects the retired agents in a symmetric way, it has an asymmetric effect on the middle-aged. With a negative shock the option to default cushions middle-aged agents from the capital loss, so that their income is higher than it would have been if they had to incur the loss. As a result the demand is higher than it otherwise would have been, implying a smaller decrease in the price of the durable and a smaller amount of consumption good available for investment. For a positive shock the wealth effect due to an increase in the price of the durable good is fully felt by the middle-aged agents, leading to an increase in demand for both the consumption and the durable good which is greater than the decrease in demand in the negative case, leading to a commensurately higher price of the consumption good.

5 Steady States and Impulse Response Functions: General Case

For general utility functions the demand of middle-aged agents depends on next period prices, while the demand of retired agents depends on previous period prices, so that the equilibrium difference equation is of order three, rather than of order one as in the previous section. In this case we restrict our analysis to a local analysis of a collateral equilibrium in the neighborhood of a steady state.

5.1 Local Dynamics Around Collateral Steady State

Consider a collateral equilibrium $\Big((c,h)_t, z_t, q_t, r_t)\Big)$ in which $z_t > 0$ so that $q_{t+1} = 1 + r_t$. We have seen in Proposition 1 that an agent's lifetime choice problem of maximizing utility subject to the sequential budget constraints (2)-(4) and the collateral constraint (5) can be transformed into two analytically simpler maximization problems, each with a single budget constraint.

Let $(c^y(e^y, q_t), h^y(e^y, q_t))$ denote the solution to the maximum problem of the young in (i) of Proposition 1, and let $(c^m(e^m, q_{t+1}, q_{t+2}), h^m(e^m, q_{t+1}, q_{t+2}), c^r(e^m, q_{t+1}, q_{t+2}), h^r(e^m, q_{t+1}, q_{t+2}))$ denote the solution to the maximum problem (ii) of the middle-aged and retired. One way of studying the equilibrium dynamics is to substitute these demand functions into equation (27) to obtain an implicit equation which must be satisfied by the equilibrium prices $(q_{t-1}, q_t, q_{t+1}, q_{t+2})$: this is how we proceeded in the previous section, and with log preferences, it led to a simple first-order equation in the price of the durable good. However in the general case the dynamics is better understood by using the durable good price and investment as the basic variables and retaining the two market clearing equations (i) and (ii) of Definition 1 at each date for the consumption and the durable good. Omitting for the moment reference in the demand functions to the endowments (which are fixed characteristics), let us define the *aggregate excess demand functions* of the three generations for the consumption and the durable goods at date t

$$\begin{aligned} F &= c^y(q_t) + c^m(q_t, q_{t+1}) + c^r(q_{t-1}, q_t) + z_t - (e^y + e^m) \\ G &= h^y(q_t) + h^m(q_t, q_{t+1}) + h^r(q_{t-1}, q_t) - z_{t-1} - (1-\delta)\left(h^y_{t-1}(q_{t-1}) + h^m_{t-1}(q_{t-1}, q_t)\right) \end{aligned} \tag{44}$$

An equilibrium is then described by a sequence of prices and investment which is a solution of the pair of equations for all t

$$\begin{aligned} F(q_{t-1}, q_t, q_{t+1}, z_{t-1}, z_t) = 0 \\ G(q_{t-1}, q_t, q_{t+1}, z_{t-1}, z_t) = 0 \end{aligned} \tag{45}$$

with appropriate initial conditions. A steady state (q, z) is a solution of the pair of equations

$$F(q, q, q, z, z) = 0, \qquad G(q, q, q, z, z) = 0 \tag{46}$$

The first equation can be viewed as a steady state "supply of investment" z as a function of q and the second as a "demand for investment" z to replace the durable good as a function of q. Let (q^*, z^*) be a solution of these equations. We know from Proposition 2 that there is at least one solution to (46), the GRC steady state: the local analysis that follows is however valid for any collateral steady state and is not restricted to the GRC.

Linearizing (45) around such a steady state (q^*, z^*) leads to the local dynamics

$$\begin{bmatrix} dq_{t+1} \\ dz_t \\ dq_t \end{bmatrix} = \Gamma \begin{bmatrix} dq_t \\ dz_{t-1} \\ dq_{t-1} \end{bmatrix} \tag{47}$$

with

$$\Gamma = \begin{bmatrix} -M^{-1}N \\ 1 \;\; 0 \;\; 0 \end{bmatrix}, \qquad M = \begin{bmatrix} F_{q_{t+1}} & F_{z_t} \\ G_{q_{t+1}} & G_{z_t} \end{bmatrix}, \qquad N = \begin{bmatrix} F_{q_t} & F_{z_{t-1}} & F_{q_{t-1}} \\ G_{q_t} & G_{z_{t-1}} & G_{q_{t-1}} \end{bmatrix}$$

where (dq_t, dz_t) denote displacements from the steady state values (q^*, z^*) and the partial derivatives in the matrices M and N are evaluated at the steady state. The dependence of F and G on investment implies $F_{z_t} = 1, G_{z_t} = 0$ so that M is triangular. If $G_{q_{t+1}} = \frac{\partial h_t^m}{\partial q_{t+1}} \neq 0$ (which eliminates the log case), M is invertible. Then

$$\Gamma = \begin{bmatrix} -\dfrac{G_{q_t}}{G_{q_{t+1}}} & \dfrac{1}{G_{q_{t+1}}} & -\dfrac{G_{q_{t-1}}}{G_{q_{t+1}}} \\ -F_{q_t} + \dfrac{F_{q_{t+1}}}{G_{q_{t+1}}} G_{q_t} & -\dfrac{F_{q_{t+1}}}{G_{q_{t+1}}} & -F_{q_{t+1}} + \dfrac{F_{q_{t+1}}}{G_{q_{t+1}}} G_{q_{t-1}} \\ 1 & 0 & 0 \end{bmatrix}$$

and the characteristic polynomial whose zeros are the eigenvalues of Γ is given by

$$G_{q_{t+1}} \lambda^3 + (G_{q_t} + F_{q_{t+1}}) \lambda^2 + (G_{q_{t-1}} + F_{q_t}) \lambda + F_{q_{t-1}} = 0 \tag{48}$$

The local stability properties of the steady state depend on the number k of roots of the characteristic polynomial lying inside the unit circle. We say that the steady state is (i) completely unstable if $k = 0$, (ii) saddle-point unstable if $k = 1$, (iii) saddle-point stable if $k = 2$, (iv) completely stable if $k = 3$. We show below that cases (i) and (iv) are unlikely to occur in this model, so the next proposition gives sufficient conditions for (ii) and (iii) to occur.

Proposition 5 (Local stability of steady state)

(i) *If* $|G_{q_t} + F_{q_{t+1}}| > |G_{q_{t+1}}| + |G_{q_{t-1}} + F_{q_t}| + |F_{q_{t-1}}|$ *the steady state is saddle-point stable.*

(ii) *If* $|G_{q_{t-1}} + F_{q_t}| > |G_{q_{t+1}}| + |G_{q_t} + F_{q_{t+1}}| + |F_{q_{t-1}}|$ *the steady state is saddle-point unstable.*

Proof: It follows from Rouche's Theorem (see e.g. Ahlfors (1979)) that if a polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 = 0$ is such that

$$|a_k| > |a_n| + \cdots + |a_{k-1}| + |a_{k+1}| + \cdots + |a_0| \tag{49}$$

i.e. if the magnitude of the coefficient of the term of order k exceeds the sum of the magnitudes of the other coefficients, then the polynomial has k roots inside the unit circle and $n - k$ outside. Condition (i) is the requirement that the coefficient of λ^2 in the characteristic polynomial (48) dominates the other coefficients, while condition (ii) is the requirement that the coefficient of λ dominates the other coefficients. □

Note that in the log case $F_{q_{t-1}} = F_{q_{t+1}} = G_{q_{t+1}} = 0$, so that (i) in Proposition 5 reduces to $|G_{q_t}| > |G_{q_{t-1}} + F_{q_t}|$, while (ii) becomes the reverse inequality. These inequalities evaluated at the GRC price $q^* = 1$ are the conditions (i) and (ii) of Proposition 3 for the stability or instability of the GRC in the log case. Thus when applied at the GRC steady state, Proposition 5 is a generalization of Proposition 3 for log preferences. However in the general case it only gives sufficient conditions, while Proposition 3 gives necessary and sufficient conditions for stability and instability in the case of log preferences.

The terms $G_{q_{t+1}}$ and $F_{q_{t-1}}$ which are the coefficients of λ^3 and λ^0 in the characteristic polynomial (48) are indirect price effects: $G_{q_{t+1}}$ indicates how the demand for the durable at date t varies with the expected price at date $t+1$, while $F_{q_{t-1}}$ indicates how the demand for the consumption good at t varies with the price of the durable at date $t-1$. These terms, which are zero with log utilities, are likely to be small in all cases compared to the direct terms G_{q_t} and F_{q_t} which indicate how the demands for the durable and consumption goods at date t vary with the current price of the durable good. If $G_{q_{t+1}}$ or $F_{q_{t-1}}$ cannot dominate the other terms, the Rouche condition (49) cannot be satisfied for $k = 3$ or $k = 0$, which suggests that a collateral steady state cannot be completely stable ($k = 3$), nor completely unstable ($k = 0$). The largest terms are likely to be the direct effects G_{q_t}, F_{q_t}, $G_{q_{t-1}}$ (where $G_{q_{t-1}}$ indicates how the demand for the durable at date $t-1$, depreciated at date t, varies with q_{t-1}) which are the only nonzero terms for the log case. Then a collateral steady state is saddle-point stable if G_{q_t} dominates the other terms, and saddle-point unstable if $G_{q_{t-1}} + F_{q_t}$ dominates the other terms.

5.2 Effect of an Unanticipated Income Shock at Date 0

Suppose that the economy has been following a steady state and that at date 0 the agents' endowments are unexpectedly shocked to $(e_0^y, e_0^m) = (1+\Delta)(e^y, e^m)$; after date 0 the endowments return to their values (e^y, e^m). We show that generically if the steady state is saddle-point stable there is a unique equilibrium path reverting to the steady state after the shock, while if the steady state is saddle-point unstable, the equilibrium path after the shock goes out of the neighborhood of the steady state, so that the impulse response analysis cannot be used.

Negative shock. We begin with the case $\Delta < 0$. The excess demand function at date 0 can be written as

$$\begin{aligned} F^0(e_0^y, e_0^m, q_0, z_0, q_1) &= c^y(e_0^y, q_0) + c^m(e_0^m, q_0, q_1) + c_0^r(I_0^r, q_0) + z_0 - (e_0^y + e_0^m) \\ G^0(e_0^y, e_0^m, q_0, q_1) &= h^y(e_0^y, q_0) + h^m(e_0^m, q_0, q_1) + h_0^r(I_0^r, q_0) - H_0 \end{aligned}$$

where

- H_0 is the supply of the durable good inherited from date -1, given by (34)
- the endowments in the demand functions of the young and middle aged are (e_0^y, e_0^m) instead of (e^y, e^m)
- the subscript '0' has been added to the demand function of the retired which differs from the general form given above since their income is different from what they had anticipated: $(c_0^r(I_0^r, q_0), h_0^r(I_0^r, q_0))$ is solution of maximizing $u(c, h)$ subject to the constraint $c + q_0 h \leq I_0^r$, where I_0^r is their unanticipated income given by (35).

At date 1 the effect of the date 0 shock to endowments still leads to excess demand functions different from F and G, since the inherited stock of the durable and the demand of the retired agents depends on the shocked endowments (e_0^y, e_0^m):

$$\begin{aligned} F^1(e_0^y, e_0^m, q_0, q_1, z_1, q_2) &= c^y(q_1) + c^m(q_1, q_2) + c^r(e_0^m, q_0, q_1) + z_1 - (e^y + e^m) \\ G^1(e_0^y, e_0^m, q_0, z_0, q_1, q_2) &= h^y(q_1) + h^m(q_1, q_2) + h^r(e_0^m, q_0, q_1) - z_0 - (1-\delta)(h^y(e_0^y, q_0) + h^m(e_0^m, q_0, q_1)) \end{aligned}$$

where we have only included endowments as arguments when they differ from the standard endowments (e^y, e^m). From date 2 on the excess demand functions are given by F and G defined by (44) and the dynamics are described by (45) with initial conditions (q_2, z_1, q_1). To determine these initial conditions we have four market clearing equations $F^0 = 0$, $G^0 = 0$, $F^1 = 0$, $G^1 = 0$ in the five unknowns $(q_0, z_0, q_1, z_1, q_2)$. The date 0 shock to endowments is taken to be sufficiently small so that the deviation of the above variables from their steady-state values can be studied using the linear approximations to the market-clearing equations. In order that the variables return to their steady-state values after date 2 following the dynamics (47), the initial condition (dq_2, dz_1, dq_1) must lie in the stable subspace spanned by the eigenvectors corresponding to the eigenvalues inside the unit circle. If the steady state is saddle-point stable (two roots inside the unit circle) the stable manifold is defined by the two associated characteristic vectors $V^1, V^2 \in \mathbb{R}^3$ and the initial conditions must satisfy

$$(dq_2, dz_1, dq_1)^\top = \nu_1 V^1 + \nu_2 V^2 \tag{50}$$

This adds three equations and two unknowns (ν_1, ν_2), so there are seven equations in the seven unknowns

$$\xi = (dq_0, dz_0, dq_1, dz_1, dq_2, \nu_1, \nu_2)$$

Generically this has a unique solution. If the steady state is saddle-point unstable (one root inside the unit circle), the stable subspace is one-dimensional and generated by a single vector $V^1 \in \mathbb{R}^3$, so the initial conditions must satisfy

$$(dq_2, dz_1, dq_1)^\top = \nu_1 V^1 \tag{51}$$

This adds 3 equations but only one unknown: there are again seven equations, but there are only 6 unknowns: with more equations than unknowns, there is generically no solution.

Assuming that the steady state is saddle-point stable, the system of equations for calculating the changes in prices and investment at dates 0 and 1 when the change in price dq_2 is compatible with convergence back to the steady state is given by

$$A\,\xi = -B\,[\,e^y, e^m]\,\Delta^- \tag{52}$$

with

$$A = \begin{bmatrix} F^0_{q_0} & F^0_{z_0} & F^0_{q_1} & 0 & 0 & 0 & 0 \\ G^0_{q_0} & G^0_{z_0} & G^0_{q_1} & 0 & 0 & 0 & 0 \\ F^1_{q_0} & F^1_{z_0} & F^1_{q_1} & F^1_{z_1} & F^1_{q_2} & 0 & 0 \\ G^1_{q_0} & G^1_{z_0} & G^1_{q_1} & G^1_{z_1} & G^1_{q_2} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & V^1_3 & V^2_3 \\ 0 & 0 & 0 & -1 & 0 & V^1_2 & V^2_2 \\ 0 & 0 & 0 & 0 & -1 & V^1_1 & V^2_1 \end{bmatrix} \qquad B = \begin{bmatrix} F^0_{e^y_0} & F^0_{e^m_0} \\ G^0_{e^y_0} & G^0_{e^m_0} \\ F^1_{e^y_0} & F^1_{e^m_0} \\ G^1_{e^y_0} & G^1_{e^m_0} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where the partial derivatives in A and B are evaluated at the steady state values $(e^y, e^m, q_0 = q_1 = q_2 = q^*, z_0 = z_1 = z^*)$. The equilibrium path $(q_t, z_t)_{t=0}^{\infty}$ following the negative shock Δ^- to the agents' endowments at date 0 and the associated interest rates $1+r_t = q_{t+1}$ can be obtained by first solving the linear equations (52) and then from date 2 on by applying the standard local dynamics (47).

Positive shock. Let Δ^+ denote the percentage increase in agents' endowments at date 0. The analysis is similar to the negative-shock case except that with a positive shock the demand of the middle-aged agents at date 0 and the retired agents at date 1 (who are the same agents) have a different form since their income consists not only of their endowment income e^m_0 but also of the capital gain on the durable good inherited from youth. The demand functions of these agents are thus based on the income

$$I^m_0 = e^m_0 + q_0((1-\delta)h^y + z^y) - b^y = e^m_0 + (q_0 - 1)b^y$$

The excess demand functions F^0 and G^0 of the negative case are replaced by the functions

$$\begin{aligned} \widetilde{F^0}(e^y_0, e^m_0, q_0, z_0, q_1)) &= c^y(e^y_0, q_0) + c^m(I^m_0, q_0, q_1) + c^r_0(I^r_0, q_0) + z_0 - (e^y_0 + e^m_0) \\ \widetilde{G^0}(e^y_0, e^m_0, q_0, q_1)) &= h^y(e^y_0, q_0) + h^m(I^m_0, q_0, q_1) + h^r_0(I^r_0, q_0) - H_0 \end{aligned}$$

and the functions F^1 and G^1 are replaced by

$$\begin{aligned} \widetilde{F^1}(e^y_0, e^m_0, q_0, q_1, z_1, q_2) &= c^y(q_1) + c^m(q_1, q_2) + c^r(I^m_0, q_0, q_1) + z_1 - (e^y + e^m) \\ \widetilde{G^1}(e^y_0, e^m_0, q_0, z_0, q_1, q_2)) &= h^y(q_1) + h^m(q_1, q_2) + h^r(I^m_0, q_0, q_1) - z_0 - (1-\delta)(h^y(e^y_0, q_0) + h^m(I^m_0, q_0, q_1)) \end{aligned}$$

The cohort demand functions which differ from those in the negative case are those of the middle aged of date 0: they no longer default, pay off their loans, and have income coming from the durable inherited from youth. This increased income increases their demand both for the consumption and the durable goods.

Proceeding as in the negative-shock case we use the market clearing equations $\widetilde{F^0} = 0, \widetilde{F^1} = 0, \widetilde{G^0} = 0, \widetilde{G^1} = 0$ and the three subspace equations (51) to determine $\xi = (dq_0, dz_0, dq_1, dz_1, dq_2, \nu_1, \nu_2)$ by the system of linear equations

$$\widetilde{A}\,\xi = -\widetilde{B}\,[\,e^y, e^m]\,\Delta^+ \tag{53}$$

where $\widetilde{A}$ and $\widetilde{B}$ are obtained from A and B by replacing the excess demand functions (F^0, F^1, G^0, G^1) by their 'tilde' versions $\widetilde{F^0}, \widetilde{F^1}, \widetilde{G^0}, \widetilde{G^1}$. From date 2 on, the deviations (dq_t, dz_t) are given by the local dynamics (47).

5.3 CES utilities

Let us apply this analysis to calculate the impulse response functions after a shock when the utility function u lies in the CES family

$$u(c,h) = \frac{1}{1-\frac{1}{\sigma}}\left(c^{1-\frac{1}{\sigma}} + \gamma\, h^{1-\frac{1}{\sigma}}\right), \qquad \sigma \neq 1$$

the limiting case where the elasticity of substitution is $\sigma = 1$ corresponding to the log case of Section 3.

Assuming that the collateral constraint binds in equilibrium so that Proposition 1 can be applied, the demand function of a typical agent in the cohort entering at date t is given by

$$\begin{aligned}
c_t^y &= \frac{e^y}{I(q_t)}, & h_t^y &= \frac{\gamma^\sigma}{(q_t-(1-\delta))^\sigma}\,\frac{e^y}{I(q_t)}\\
c_{t+1}^m &= \frac{e^m}{J(q_{t+1},q_{t+2})}, & h_{t+1}^m &= \frac{\gamma^\sigma}{(q_{t+1}-(1-\delta))^\sigma}\,\frac{e^m}{J(q_{t+1},q_{t+2})}\\
c_{t+2}^r &= \frac{\beta^\sigma (q_{t+2})^\sigma e^m}{J(q_{t+1},q_{t+2})}, & h_{t+2}^r &= \frac{(\beta\gamma)^\sigma e^m}{J(q_{t+1},q_{t+2})}
\end{aligned} \tag{54}$$

with

$$I(q_t) = 1 + \gamma^\sigma (q_t - (1-\delta))^{1-\sigma}, \qquad J(q_{t+1},q_{t+2}) = I(q_{t+1}) + \frac{\beta^\sigma}{(q_{t+2})^{1-\sigma}} + (\beta\gamma)^\sigma \tag{55}$$

where we have used the relation $1 + r_{t+1} = q_{t+2}$ which holds in an equilibrium with positive investment and thus in a steady state equilibrium. Of particular interest is the GRC steady state with prices $q_t = 1$ for $t \in \mathbb{Z}$. Evaluating the demands (54) at the prices $q_t = 1$ for $t \in \mathbb{Z}$ gives the value $(c,h) = (c^y, h^y, c^m,\, h^m, c^r, h^r)$ of the agents' consumption at the GRC, from which it is easy to deduce the financial variables (b^y, b^m, z^y, z^m). In order that the borrowing constraint be binding for the young agents at the steady state we must have $u_c(c^y, h^y) > \beta\, u_c(c^m, h^m)$ which is equivalent to

$$e^y \leq \frac{(1+\gamma^\sigma \delta^{1-\sigma})\, e^m}{\beta^\sigma \Big(1+\beta^\sigma + \gamma^\sigma \delta^{1-\sigma} + (\beta\gamma)^\sigma\Big)} \tag{56}$$

(56) reduces to (29) when $\sigma = 1$ and, as in Section 3, we restrict the analysis to parameter values satisfying this condition.

Numerically computing the solutions of the steady state equations (46) reveals that the property found for the log case, namely that there are two steady states, the GRC and a second steady state with no intermediation, carries over to the CES family when the elasticity of substitution is above a critical value $\sigma \geq \sigma^*$, while for low elasticities $\sigma < \sigma^*$ the no-intermediation steady state disappears, and the GRC is the unique steady state. σ^* depends on the other parameters $(\beta, \gamma, \delta, e^y, e^m)$ and is less than one since the properties for $\sigma \geq \sigma^*$ are those for the log case.

Since we have an explicit expression for the demand functions, the condition in (i) of Proposition 5 which ensures that the GRC is saddle-point stable can be expressed as a condition on the parameters $(\beta, \delta, \gamma, \sigma, e^y, e^m)$: this however gives a less tractable expression than (i) in Proposition 3, to which it reduces when $\sigma = 1$. When $\sigma < 1$ it is satisfied for most values of the parameters except when γ is close to zero; when $\sigma > 1$ it requires that γ/δ be sufficiently large, expressing the requirement that the durable good is sufficiently desirable and durable. The reason why the stability condition (i) in Proposition 5 holds for most parameter values when $\sigma < 1$ is that, given the lack of substitutability, the price effect G_{q_t} on the demand for the durable good is larger than the price effect F_{q_t} on the demand for the consumption good, so that G_{q_t} dominates the term $F_{q_t} + G_{q_{t+1}}$, which is the dominant term on the right hand side of the inequality (i).

To calculate response functions of the prices and investment to an anticipated shock at date 0 we need to choose reference values for the parameters $(e^y, e^m, \beta, \gamma, \delta, \sigma)$. As in our earlier paper (Geanakoplos-Magill-Quinzii (2004)) we have taken the economic life of an agent to last for three periods, young, middle age and retirement. If childhood is included as the 'non economic' part of an agent's life, and if the life span is 80 years then each period corresponds to 20 years. What matters for the agents' endowments is not their magnitude but the relative magnitude e^y/e^m. To reflect the fact that middle-aged agents are more productive we set $(e^y, e^m) = (2, 5)$. We choose $\beta = 0.7$ corresponding to an annual discount rate of 2%. We think of the durable good as housing and choose $\delta = 0.3$, implying that after 20 years 1/3 of a house needs to be replaced to maintain its original condition. If a young agent with CES preferences spends a proportion π of his income on the durable good and $1-\pi$ on the consumption good at the GRC, then $\delta\Big(\frac{\gamma}{\delta}\Big)^\sigma) = \frac{\pi}{1-\pi}$. To express the condition that the durable good is ‘desirable’ for the agent we assume that $\pi \geq 1/4$ so that

$$\delta \left(\frac{\gamma}{\delta}\right)^{\sigma} \geq \frac{1/4}{3/4} = \frac{1}{3} \tag{57}$$

Since the elasticity of substitution affects the relative sizes of the effects on prices and investment, we present two cases, $\sigma = 1/3$ and $\sigma = 3$. For $\sigma = 1/3$ the inequality (57) implies $\gamma \geq 0.41$ and for $\sigma = 3$ it implies $\gamma \geq 0.31$: we choose $\gamma = 0.5$ which is compatible with both cases. To remove the indeterminacy between z^y and z^m we choose $z^y = 0$.

Figures 3 and 4 show the deviations of the durable good prices (dq_t) and investment (dz_t) from the GR steady state following unanticipated negative and positive shocks of -10% and +10% to the agents’ endowments at date 0, for the reference values of the parameters and an elasticity of substitution of $\sigma = \frac{1}{3}$. Figures 5 and 6 show the impulse response functions for the same values of the parameters and $\sigma = 3$. The curves IRF_q and IRF_z—the impulse response functions to the date 0 shock—have a superscript $-$ and $+$ when they represent the response of prices and investment to a negative and positive shock respectively. The dotted curve $symIRF^+$ is the symmetric image of IRF^+ with respect to the steady-state value (it graphs the values of $q - dq_t^+$ in Figures 3 and 5 and the value of $z - dz_t^+$ in Figures 4 and 6) which shows the asymmetry between the response to the negative and positive shock. The price at date 0 responds more strongly to a positive than a negative shock ($symIRF_q^+$ below IRF_q^-) while the investment responds more strongly (IRF_z^- below $symIRF_z^+$). Because the decrease in investment at date 0 when a negative shock occurs creates a low supply of the durable good at date 1 the price rebounds strongly, overshooting the steady state. Conversely the increase in investment following a positive shock depresses the price of the durable at date 1 below the steady state price. However since investment responds less strongly to a positive than a negative shock the fall in price is relatively less pronounced than the rise in the negative case (IRF_q^- stays below $symIRF_q^+$ at date 1). As could be predicted, the response of the durable good price to an income shock is much larger when the elasticity of substitution is small ($\sigma = 1/3$) than when it is large ($\sigma = 3$): in the latter case the response of investment is somewhat larger since the demand for the consumption good varies more with the change in income.

6 Conclusion

Studying an economy which is in a steady state equilibrium and subjecting it to an unanticipated shock provides a useful analytical tool for examining how default affects equilibrium prices and investment. The OLG model provides a setting where borrowing and saving is part of an agent’s

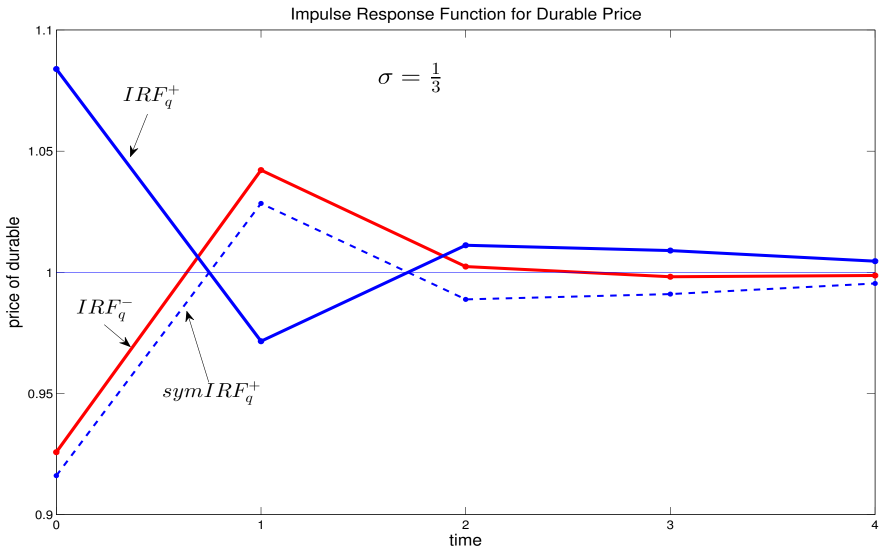

Figure 3: Impulse response function of the durable good price in the case of a negative shock (IRF_q^-), of a positive shock (IRF_q^+), and the symmetric image of the latter with respect to the steady state line ($symIRF_q^+$) for $\sigma = 1/3$.

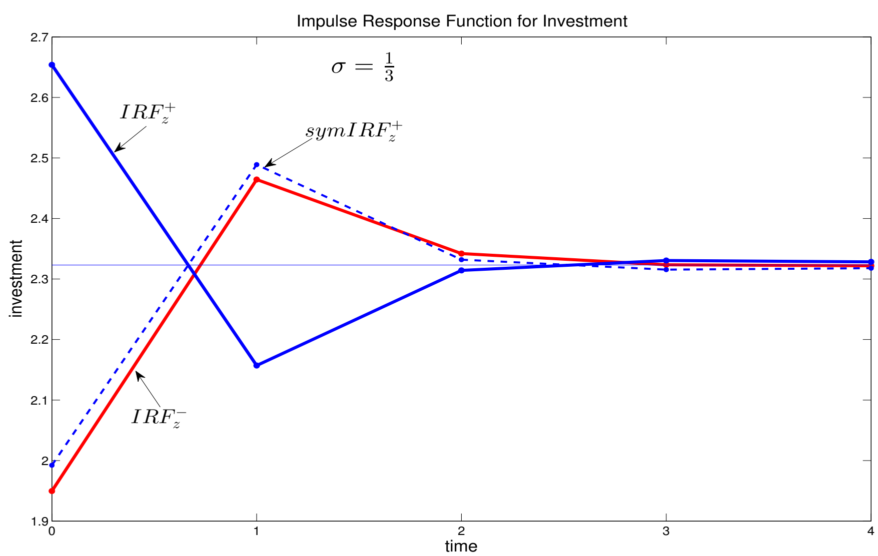

Figure 4: Impulse response function of investment in the case of a negative shock (IRF_z^-), of a positive shock (IRF_z^+), and the symmetric image of the latter with respect to the steady state line ($symIRF_z^+$) for $\sigma = 1/3$.

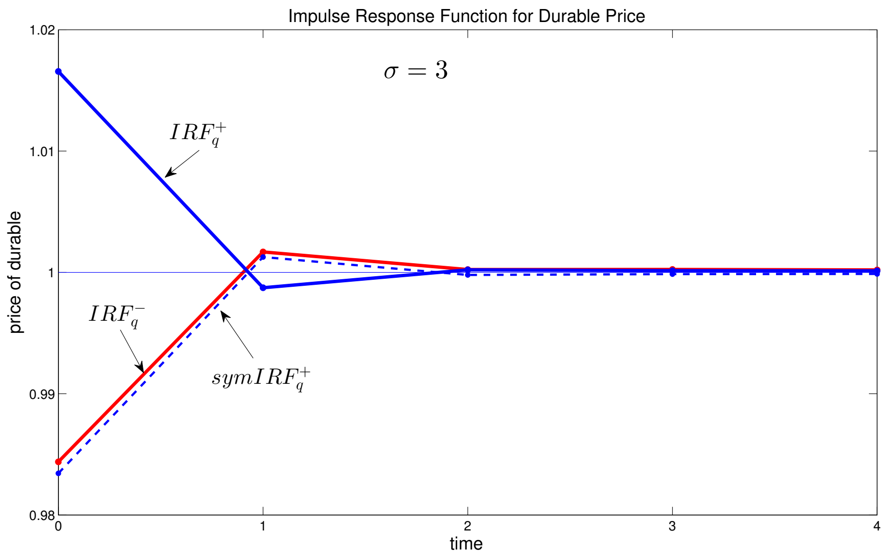

Figure 5: Impulse response function of the durable good price in the case of a negative shock (IRF_q^-), of a positive shock (IRF_q^+), and the symmetric image of the latter with respect to the steady state line ($symIRF_q^+$) for $\sigma = 3$.

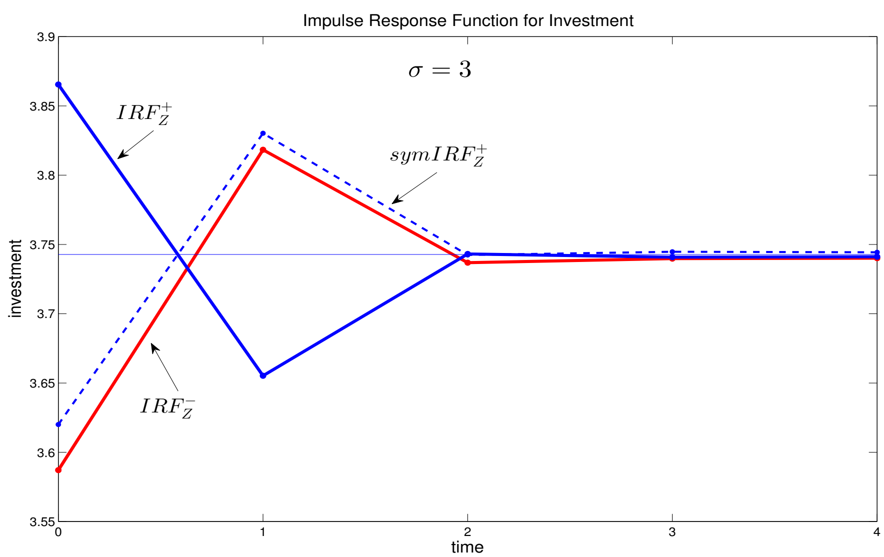

Figure 6: Impulse response function of investment in the case of a negative shock (IRF_z^-), of a positive shock (IRF_z^+), and the symmetric image of the latter with respect to the steady state line ($symIRF_z^+$) for $\sigma = 3$.

life-cycle behavior: to this we added the realistic feature, introduced in the GE literature (Dubey et al. (1995)) that agents must back their loans by holding a durable good which serves as collateral. In contrast to most models with infinite-lived agents studied in macroeconomics, the OLG model typically has several steady states whose stability properties depend on the parameters, in particular the lifetime income profile of the agents. Despite the richer structure of our model—three-period lived agents, two goods, one perishable, one durable, with production of the durable from the consumption good—we are able to extend the results of the two-period lived OLG exchange or Diamond model with production to a surprising degree. In particular we show that the equivalent of the Golden Rule—the Golden Rule with Collateral (GRC)—is the natural steady state around which to analyze the impact of an unanticipated shock because, in contrast to the perishable good case, it is the 'most stable' steady state. For the presence of the durable good, which not only provides agents utility services but also provides an instrument for transferring income to their retirement, significantly alters the long run behavior of the economy: while in the model with a perishable good the Golden Rule is unstable, it becomes stable when agents can save by investing in the durable good. It seems that this stability property of the GRC, which in this paper has been a by-product of introducing a durable good which serves as collateral, can provide an interesting subject for future research.

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