

## CONTENTS

1. <b>Introduction</b>	<b>1</b>
2. <b>General setting and assumptions</b>	<b>5</b>
3. <b>Choosing between two investment projects: an extension of Wicksell's problem</b>	<b>7</b>
4. <b>Choosing between alternative replacements repeatedly: an extension of Faustman's problem</b>	<b>10</b>
5. <b>Choosing between alternative replacements repeatedly: the solution</b>	<b>17</b>
<b>5.1 Numerical resolution</b> . . . . .	<b>24</b>
6. <b>Conclusion</b>	<b>28</b>
7. <b>APPENDIX</b>	<b>33</b>

Alternative and Indefinitely Repeated Investments: Species Choice and  
Harvest Age in Forestry<sup>1</sup>

**Skander Ben Abdallah**  
*Université du Québec à Montréal*  
ben\_abdallah.skander@uqam.ca

**Pierre Lasserre**  
*Université du Québec à Montréal,*  
CIREQ, and CIRANO  
lasserre.pierre@uqam.ca

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## Abstract

The tree harvest problem of forest management is an archetypal investment problem; it involves time, uncertainty, and irreversible actions with consequences in the future. The exercise of the option to cut a tree opens the option of planting a new one or of using the land for alternative purposes. We enrich the tree harvest problem by considering the planting decision too with no cost associated with harvesting or planting. Two tree species are available; their growth is deterministic but their timber unit price is stochastic. In the case of a single rotation, known as the Wicksellian tree harvest problem, the forest manager should plant one species immediately if its price is sufficiently high relative to the price of the other species. However, if prices are close to each other, the manager should wait in order to avoid the mistake of planting the wrong species. Waiting should last until the prices are sufficiently far apart to make the probability of a future price reversal acceptably low. In contrast, when the number of rotations is arbitrarily high, as in Faustman's problem, waiting before planting the new tree, whatever its species, is never optimal once a harvest has taken place. However, the optimal harvest age depends on the relative timber price. We show that the optimum harvest age increases when the relative price approaches a threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species; letting existing trees grow older is a way to postpone the choice. We also show that the land value increases with the uncertainty of timber prices. The stand value is similar to the value of an American option with a free boundary and an indefinite expiry date but with endogenous payoff.

*key words:* Real options; stochastic prices; American options; free boundary; penalty method; forestry; Faustman; alternative species; rotation.

*J.E.L. classification:* C61; D81; G11; G13; Q23.

## Résumé

La décision de couper un arbre en gestion forestière est l'archétype du problème d'investissement; il implique le temps, l'incertitude ainsi que l'irréversibilité d'actions ayant des conséquences dans le futur. L'exercice de l'option de couper un arbre donne naissance à l'option de planter un nouvel arbre ou bien d'utiliser la terre à d'autres fins. On enrichit ce problème en considérant aussi la décision de planter. Deux essences d'arbres sont disponibles; leurs fonctions de croissance sont déterministes mais les prix de leurs bois sont stochastiques. Lorsqu'une seule rotation est envisagée, il s'agit du problème dit de Wicksell; le gestionnaire forestier plante immédiatement une essence si le prix de son bois est suffisamment élevé relativement au prix de l'autre essence. Cependant, si les deux prix sont proches, le gestionnaire doit attendre dans le but de ne pas commettre l'erreur de choisir la mauvaise essence. L'attente doit se prolonger jusqu'à ce que les prix se distinguent suffisamment pour que la probabilité que l'espèce choisie ne soit plus l'espèce désirée ultérieurement devienne suffisamment faible. En revanche, dans le problème de Faustman, où le nombre de rotations est illimité, il n'est jamais optimal d'attendre avant de planter une essence une fois qu'une récolte a eu lieu. Cependant, l'âge des arbres à la récolte dépend du prix relatif de leur bois. On montre que l'âge optimal à la récolte croît quand le prix relatif est suffisamment proche d'un seuil signalant la nécessité de planter l'essence alternative. Au lieu d'attendre en laissant le terrain nu lorsqu'il n'est pas suffisamment certain de l'espèce à choisir, le gestionnaire laisse croître plus longtemps les arbres en place, ce qui est une manière de reporter la décision. La valeur d'un terrain planté est similaire à la valeur d'une option américaine avec frontière libre, échéance infinie et paiement endogène.

*mots clés:* Options réelles; prix stochastiques; options américaines; frontière libre; méthode de pénalité; essences alternatives; rotation.

*Qualification JEL:* C61, D81, G11, G13, Q23.

## 1. INTRODUCTION

Forest management is an archetypal investment problem; it involves time; it involves uncertainty; it involves irreversible actions with consequences in the future. It also exemplifies investments that open up new options: cutting a tree opens the option of planting a new one or using the land differently. Faustman (1849) gave forestry economics its foundations by addressing the question: at what age should a stand of even-aged trees be harvested? He did so under the assumption of constant timber prices by comparing the net marginal benefits from letting timber grow further, to the opportunity cost of existing trees plus the opportunity cost of the land, itself a function of timber management decisions.

Faustman's original problem has been refined and generalized in many ways. In this paper we focus on the availability of alternative species to replace the trees being harvested. We show how this choice should be made and timed, and how it affects the harvesting decision. While species choice is clearly important for the forester, that question has much wider relevance and may be viewed as illustrating a general investment problem. The question of choosing a harvest age is akin to that of deciding when an equipment should be retired; the species choice is similar to the choice of alternative technologies for replacement. Thus we introduce alternative projects in situations where decisions open up new options in a process that repeats itself indefinitely, and must be reevaluated at each instant. This most common practical situation has not been investigated theoretically before, although both the real option literature and the forestry literature have gone some way toward solving that problem.

The real option approach has been widely applied in natural resources exploitation and management. In the real option framework, a typical investment involves an optimum stopping rule, the choice of the date at which the decision maker considers that conditions have become favorable enough to justify committing resources irreversibly. A typical result is that more uncertainty postpones investment although it increases project value.

Applying this approach, a number of studies (Brazee and Mendelsohn (1988), Clarke and Reed, 1989; Thomson, 1992; Insley, 2002) examine the optimal harvesting age in forestry under stochastic timber price.<sup>2</sup> These studies show that when timber prices or stand value follow a Brownian motion and in absence of management costs, the solution is similar to the deterministic case. However, when management costs are considered, the expected value of the stand is increased by considering a reservation price policy that exploits stochastic variations in prices. Uncertainty provides an incentive to postpone harvest and delay management costs until observing future prices and schedule harvests accordingly. Platinga (1998) shows that the option value to delay harvest when timber prices are stationary stems both from the level of the current stumpage price relative to its long-term mean and from the stand value relative to the fixed management cost. Over time, applications have been extended to include more and more problems, such as differentiated timber prices (Forboseh et al., 1996), uneven-aged management (Haight, 1990), multi-species stands under changing growth conditions caused by climate change (Jacobsen and Thorsen, 2003).

The real option literature has begun to treat situations where an action of the decision maker simultaneously involves a choice between alternative opportunities and choosing the timing of an investment. These alternative opportunities may differ with respect to investment costs and output as in Decamps et al (2006) or benefit trajectories as in Kassari and Lasserre (2004). An interesting result in such situations is that a new reason for postponing action arises. When the alternatives are too close to each other and uncertain, the decision maker may choose to wait in order to avoid choosing an alternative that might prove to be less desirable than another candidate in the future. This inaction may be optimal although each project, taken in isolation, would satisfy the requirements for immediate investment under conditions of irreversibility and uncertainty. Although the forestry literature has considered choices between alternative options (Reed, 1993; Conrad, 1997; Abildtrup and Strange, 1999) such as the investment problem of stand establishment, no attention has been devoted to the hysteresis possibly induced in such

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<sup>2</sup>Willassen (1998) dismisses the optimal stopping methodology and uses impulse control.

circumstances. Furthermore, when simultaneous alternatives were considered, the future stand value was treated as exogenous, independent of the current choice. For instance, Thorsen (1999a) analyses the choice of tree species for afforestation as a real option problem, and Thorsen and Malchow-Møller (2003a) extend it to a two-option problem with two mutually exclusive options (two tree species), where exercising one option implies losing the other. Jacobsen (2007) goes one step further: upon harvest, the current stand (of spruce) may be allowed to regenerate naturally and costlessly, or may be replaced with oaks at some cost. However, it is assumed that at some finite time horizon oaks will be definitely planted so that the problem eventually simplifies to the indefinitely repeated single-species problem of Faustman, with stochastic price. Finally, it is not certain whether the age at which harvest is optimal is higher or lower than Faustman's or Wicksell's rotation.

To our knowledge our paper is the first one involving an indefinitely repeated choice between options. Given an existing stand composed of one of two possible species, the decision maker chooses the harvest age. Then, she decides which species or which proportion of species should be used to repopulate the land, where timber prices evolve stochastically, possibly with some correlation. Then again, and forever, harvests times and species choices must be decided optimally. To draw again on the analogy with equipment retirement and replacement, equipment may be retired at any age and replaced with any of two alternative technologies where the efficiency of each technology evolves stochastically as suppliers improve their products. It is not certain that one type will dominate the other for long, let alone forever.

The general setting and assumptions are introduced in Section 2. After harvesting, the land may be repopulated with any combination of two tree species; the new crop may be established right after the harvest or at any later time. Each species is characterized by a different, stochastic, timber price process and by a distinct, although not stochastic, growth function. In Section 3, we investigate the case of a single rotation, also known as Wicksell's tree harvest problem already analyzed for a single species by Willassen (1998). With two species, one should consider the species choice and its timing, before

considering harvest. A form of hysteresis not previously identified in the literature arises: under some conditions about relative species prices, the decision maker waits rather than establishing a crop, despite the fact that each species would be worth planting in isolation.

In Section 4 we extend the analysis to multiple rotations. The decision maker must decide at what age the trees of the current stand must be cut; after harvest she must decide whether she should establish a new crop or wait; when she makes such decision, she must choose one, or the other, species. This process is repeated indefinitely. We find that the hysteresis of the Wicksellian problem disappears. The qualitative properties of the decision rules and value functions are described analytically and presented in a number of propositions.<sup>3</sup> The stand value is similar to the value of an American option with a free boundary and an expiry date equal to infinity but with endogenous payoff. Land value increases with the volatility of timber prices. It is continuous and differentiable even around the switching threshold. . The optimum harvest age increases when the relative price approaches some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species; letting existing trees grow older is a way to postpone the choice, allowing the prices to become more informative. Conversely, if the relative price exceeds the switching threshold, the risk of making a mistake by switching diminishes as the relative price moves further away from the threshold, so that the reason to postpone harvest becomes less compelling and the optimum harvest age diminishes. This possibility to postpone harvest explains why the hysteresis observed in the Wicksellian problem, taking the form of forest land being left bare until a decision to plant is made, disappears when harvests can be indefinitely repeated.

Before concluding in Section 5, we complete the analysis of Section 4 with a numerical resolution based on the penalty method (Zvan et al 1998) applied simultaneously to the

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<sup>3</sup>The assumption that prices follow geometric Brownian motions helps obtain some of the analytical results. However the numerical treatment does not depend on that assumption and the qualitative properties of the solution, as well as the intuitive understanding arising from the analytical results still stand to changes in it.



stand value function of each species, and on a Newton iterative process applied to the land value. This allows explicit comparisons with the non stochastic solution of the model obtained entirely analytically and presented in the Appendix.

Section 5 concludes.

## 2. GENERAL SETTING AND ASSUMPTIONS

We study the decision by a risk-neutral decision maker (DM) to establish one, or any mix, of two alternative tree species  $P$  and  $P'$  on a plot of bare land. The timber price of species  $P$  (respectively  $P'$ ) follows a geometric Brownian motion (GBM) with drift  $\mu$  (resp.  $\mu'$ ) and volatility  $\sigma$  (resp.  $\sigma'$ ):

$$dp = \mu p dt + \sigma p dz \tag{1a}$$

$$dp' = \mu' p' dt + \sigma' p' dz \tag{1b}$$

where time indices have been omitted,  $dz = \varepsilon \sqrt{dt}$  and  $dz' = \varepsilon' \sqrt{dt}$  are the increments of Wiener processes, and  $\varepsilon$  and  $\varepsilon'$  are standardized Gaussian white noises whose correlation is  $\rho$ . In the rest of the paper variables that depend on time are indexed unless they are considered at the current date defined as  $t = 0$ , in which case the time index is omitted. The relative price  $\theta = \frac{p'}{p}$  is time variable while  $\delta = r - \mu > 0$  and  $\delta' = r - \mu' > 0$ <sup>4</sup> are constant parameters assumed to be positive, where  $r$  is the discount rate. Thus the price of one species may be lower or higher than the price of the other species at anyone time and that situation may reverse itself any number of times. When  $\mu = \mu'$  reversals are purely random events; when one of the drifts is higher than the other one the probability that the high-drift species price-dominate the other one increases over time, although the possibility of a reversal only vanishes in the limit when the relative price tends toward infinity. If both volatilities are zero and  $\mu = \mu'$ , the relative prices remain constant so that one species should be preferred to the other one forever. If both volatilities are zero and one drift is higher than the other one, it is certain that the high-drift species will override the other one at some known time if its harvest value is not higher already.<sup>5</sup>

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<sup>4</sup>We assume that  $\delta > 0$  and  $\delta' > 0$ ; otherwise it would be optimal to hold on to the trees forever.

<sup>5</sup>Taking into account the species growth function.

Consequently that species has to be chosen whether immediately or at some known date in the future.

Each tree species is characterized by a timber volume growth function with the following properties:

**Assumption 1** *There exists  $\underline{a} > 0$  and  $\underline{a}' > 0$ , such that the timber volume functions  $V(a)$  and  $V'(a)$  are continuous over  $[0, +\infty[$ ,  $V(a) = 0$  over  $[0, \underline{a}]$ ,  $V'(a) = 0$  over  $[0, \underline{a}']$ ;  $V(a)$  and  $V'(a)$  are positive, continuous, differentiable and concave over  $[\underline{a}, +\infty[$  and  $[\underline{a}', +\infty[$  respectively. In addition,  $\lim_{a \rightarrow +\infty} V_a(a) = 0$  and  $\lim_{a \rightarrow +\infty} V'_a(a) = 0$ .*

Empirical volume growth functions have a convex initial part and become concave once the trees have reached some strictly positive age. This implies that it is never optimum to harvest at an arbitrarily low age if the trees are the sole source of value for the land tract. Assumption 1 ensures that this stylized property is satisfied while avoiding delicate and economically trivial complications associated with the non convexity of the volume functions at low tree ages<sup>6</sup>. Similarly, we assume that operational costs (while trees are growing) and harvesting costs are either nil or accounted for in prices  $p$  and  $p'$ .

A more consequential assumption is that planting and other initial investment costs are zero. We make that assumption for two reasons. The first one is theoretical. A key element of real options decisions is the irreversible commitment of resources at the time a project is undertaken. When these resources take the form of an irreversible investment cost, and the future returns of the project are uncertain, they create a deferral option or an option to wait that is well understood.<sup>7</sup> This paper focuses on the option to choose, once or an indefinite number of times, between two alternatives; this option is best identified in the absence of sunk costs, an assumption that eliminates the conventional deferral option. The second reason for eliminating investment costs is that this assumption allows to go further into the analytical resolution of the problems to

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<sup>6</sup>In the numerical illustrations, we use the same volume growth function for both species; this better isolates the role of the stochastic price processes defined by (1a) and (1b). The volume growth function used is  $V(a) = V_\infty (1 - e^{-\alpha(a-\underline{a})})$  where  $V_\infty = 100$  is the timber volume when the age tends to infinity,  $\alpha = 0.01$ , and  $\underline{a} = 10$  is the minimum age for positive growth.

<sup>7</sup>See the references in the introduction.

describe and understand the impact of the switching option on the investment decision. The numerical method developed to illustrate the solution and to complement the results can be adapted to a model involving a positive planting or investment cost.

### 3. CHOOSING BETWEEN TWO INVESTMENT PROJECTS: AN EXTENSION OF WICKSELL'S PROBLEM

"The Wicksellian tree cutting problem" refers to the problem of choosing the age at which a stand of even-aged trees will be harvested. One single harvest is considered and the land is assumed to be of negligible value beyond this unique harvest. The optimal harvesting age is determined by the well-known Wicksellian rule under which the optimal age is chosen in such a way that the marginal value growth of the trees is equal to the opportunity cost of holding them.

We modify the stochastic version of Wicksell's problem in two essential ways. First we start with bare land and consider the tree planting decision. Second we study a situation where two tree species, not simply one, are available. The reasons why these new features are important are the following. First, since the timber value of each species evolves stochastically, the revenues derived upon harvest depend on the species initially planted and on the price reached at harvesting. The harvest value of one species may overtake the other one, implying that the decision maker may regret the initial choice. Second, the availability of more than one species opens up the possibility of diversification that we show is not optimal in the following proposition.

**Proposition 1** *When two tree species may be grown simultaneously on a forest land, it is optimal to specialize into one, or the other, species rather than diversify.*

**Proof.** See the appendix. ■

Proposition 1 indicates that the stand value at planting time is highest when one single species, rather than a combination of the two species, is established on the plot. Clearly, however, a combination with lower expected value but less risk might be preferable if the decision maker was risk averse. Let  $G(p, a)$  ( $G'(p', a)$ ) be the stand value

function when species  $P$  (resp. species  $P$ ) is in place, the age of the trees is  $a$ , and the current price of the species in place is  $p$  (resp.  $p'$ ):

$$G(p, a) = \max_s E [e^{-rs} V(a+s)p_s] \quad (2a)$$

$$G'(p', a) = \max_s E [e^{-rs} V'(a+s)p'_s] \quad (2b)$$

Consider  $G(p, a)$ . As  $G(p, a)$  is homogenous in  $p$ ,  $G(p, a) = pg(a)$  where  $g(a) \equiv G(1, a)$ .  $G(p, a)$  must satisfy Bellman's equation  $EdG = rGda$  which implies, by Itô's lemma, that  $g(a)$  satisfies  $\delta g(a) - g_a(a) = 0$ . Then  $g(a) = be^{\delta a}$ , where  $b$  is a constant to be determined using the value-matching and smooth-pasting conditions as follows. At harvest age  $a_w$  and for any price  $p$ ,  $G(p, a_w) = V(a_w)p$ ,  $G_p(p, a_w) = V(a_w)$ , and  $G_a(p, a_w) = V_a(a_w)p$ , implying  $g(a_w) = V(a_w)$  and  $g_a(a_w) = V_a(a_w)$ . Consequently,

$$G(p, a) = be^{\delta a} p \quad (3a)$$

$$b = V(a_w) e^{-\delta a_w} \quad (3b)$$

$$\frac{V_a(a_w)}{V(a_w)} = \delta \quad (3c)$$

where  $a_w$  is determined implicitly by the last equation as a function of  $\delta$  independent of price. In particular when the expected value of timber is constant ( $\mu = 0$ ), the optimum harvest age is given by condition  $\frac{V_a(a_w)}{V(a_w)} = r$ , which is Wicksell's rule in the absence of uncertainty (Willassen, 1998).

To examine the effect of the existence of the second tree species, note similarly that

$$G'(p', a) = b' e^{\delta' a} p' \quad (4a)$$

$$b' = V'(a'_w) e^{-\delta' a'_w} \quad (4b)$$

$$\frac{V'_a(a'_w)}{V'(a'_w)} = \delta' \quad (4c)$$

In particular, when the age of the trees is zero, the stand value is either  $bp$  or  $b'p'$ , depending on the species. If the species is yet to be chosen, the stand value is  $\mathbf{xx} \max_T [be^{-rT} E p_T, b'e^{-rT} E p'_T]$ , where  $T$  is the date at which the chosen species will be established. As only one harvest is possible in Wicksell's problem and the land has no

value further to that harvest, this is also the value of land xx and therefore:

$$F(p, p') = \max_T \left[ b e^{-\delta T} p, b' e^{-\delta' T} p' \right] \quad (5)$$

Since  $F(p, p')$  is homogenous in  $(p, p')$  it can be written  $F(p, p') = p f(\theta)$  where  $\theta = \frac{p'}{p}$ .

Suppose that the land is currently bare and that it is optimal to wait rather than establish a new crop at the current time. Then there exists a time interval  $dt$  during which  $F(p, p')$  satisfies Bellman's Equation  $EdF = rFdt$ . This defines the continuation region in the plan  $(p, p')$ . Itô's lemma implies that  $f(\theta)$  satisfies:

$$\frac{\bar{\sigma}^2}{2} \theta^2 f_{\theta\theta}(\theta) + \bar{\mu} \theta f_{\theta}(\theta) - \delta f(\theta) = 0 \quad (6)$$

with  $\bar{\sigma}^2 \equiv \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$  and  $\bar{\mu} \equiv \mu' - \mu = \delta - \delta'$ .

This differential equation has solutions (*i.e.* the continuation region exists) when its characteristic equation  $Q(\beta) = 0$  has real roots, where  $Q(\beta) = \frac{\bar{\sigma}^2}{2}\beta(\beta - 1) + \bar{\mu}\beta - \delta$ . This requires the equation to be quadratic and its determinant  $\Delta \equiv (\bar{\mu} - \frac{\bar{\sigma}^2}{2})^2 + 2\bar{\sigma}^2\delta$  to be non negative. The equation is quadratic if and only if either  $\sigma > 0$  or  $\sigma' > 0$ . If this condition does not hold, the species yielding the highest return at harvest is known for any crop establishment time. Since  $\mu < r$  and  $\mu' < r$ , it is then more profitable to establish the stand immediately than later. Consequently it is never optimal to wait before choosing a species if  $\sigma = \sigma' = 0$ . The determinant of the characteristic equation is zero if and only if  $(\mu = \mu', \sigma = \sigma', \rho = 1)$  or if  $(\mu = \mu', \sigma = \sigma' = 0)$ . In such cases, prices follow parallel trajectories, so that the higher price will remain so forever; it is then optimal to invest immediately in the project with the highest price. To rule out these trivial cases where the continuation region does not exist, we make the following assumption:

**Assumption 2** *The future price of at least one species is uncertain i.e.  $\sigma > 0$  or  $\sigma' > 0$  and  $(\mu, \sigma) \neq (\mu', \sigma')$ .*

Under Assumption 2, the characteristic equation has two roots  $\beta_1 > 1$  and  $\beta_2 < 0$ . In that case,

$$f(\theta) = b_1 \theta^{\beta_1} + b_2 \theta^{\beta_2} \quad (7)$$

where  $b_1$  and  $b_2$  are two constants. The option value function is then  $F(p, p') = pf(\theta)$  in the continuation region. However, at some levels of  $\theta$ , it is optimal to invest rather than wait. Precisely,

**Proposition 2** (*one rotation*) *When one of two tree species must be established for a single harvest, the optimal decision is to leave the land bare unless the relative timber price is out of an interval  $[\theta^*, \theta'^*]$ . The first time that the relative price reaches  $\theta'^*$  from below (resp.  $\theta^*$  from above) species  $P'$  (resp.  $P$ ) should be established. Once species  $P$  (resp.  $P'$ ) has been established the stand should be allowed to grow until it reaches its optimal harvest age defined by Formula (3c) (resp. (4c)), which is independent of the relative price. As long as the land is bare, its value is  $pf(\theta)$  given by (7); once populated with age  $a$  trees of species  $P$  (resp.  $P'$ ), the land is worth  $be^{\delta a}p$  (resp.  $b'e^{\delta a}p'$ ) where  $b$  is given by (3b) (resp.  $b'$  is given by (4b)).*

**Proof.** See the Appendix for details not in the main text. ■

The decision to go ahead with the investment is postponed until timber prices have differentiated themselves clearly enough. This hysteresis leaves the land bare despite the fact that it would be unambiguously optimal to plant any of the two species immediately if the option to establish the other one was not available. In situations where the returns expected from a project, while high enough to warrant investment in isolation, might in the future be overtaken by the returns from an alternative project, using such a decision rule reduces the probability of finding out *ex post* that the least profitable option was chosen, irreversibly. Clearly the hysteresis may occur under any parameter combination satisfying Assumption 2, whether the drifts of the price processes are identical or not, and whether or not the price of the chosen species is subject to uncertainty.

#### 4. CHOOSING BETWEEN ALTERNATIVE REPLACEMENTS REPEATEDLY: AN EXTENSION OF FAUSTMAN'S PROBLEM

In order to keep the notation simple, some functions used in Section 3. will be redefined to account for multiple rotations; otherwise the notation is unchanged. Multiple harvests

is the realm of Faustmann and his followers, who initially considered successive harvests of a single species whose price was both certain and constant. The problem is to find the age at which it is optimal to cut the trees. It has been shown that the optimal harvest age may be independent of the price, whether that price is stochastic or constant.

When in addition two species are available and their prices are stochastic, we will show that the optimum age depends on the species in place and on the relative price of both species. Furthermore, we will treat separately the decision to cut and the decision to establish the same or the alternative species. We start with existing results when there is one single species.

When only one species  $P$  is available, the expected value of harvesting after some time  $s$  a forest stand whose current age is  $a$ , and then replanting and harvesting the same species for an arbitrary number of rotations, is

$$G(p, a) = \max_s E e^{-rs} [p_s V(a + s) + F(p_s)] \quad (8)$$

where  $F(p)$ , the land value, solely depends on the current price of the single species available:

$$F(p) = \max_s E e^{-rs} [p_s V(s) + F(p_s)] \quad (9)$$

Thus, the land value coincides with the stand value at age zero as implied by the assumption that there are no planting or other investment costs; that is  $F(p) = G(p, 0)$ .

When the timber price is constant, the optimal harvesting age  $a_F$  is constant from one rotation to the next and determined by Faustman's rule (1849); it is independent of the timber price and implicitly given by:

$$\frac{V_a(a_F)}{V(a_F)} = \frac{r}{1 - e^{-ra_F}} \quad (10)$$

The analysis has been extended to situations where the price may change over time (Brazeel and Mendelsohn, 1988; Clarke and Reed, 1989; Morck et al. 1989; Thomson, 1992; Reed, 1993). When the price is governed by (1a) and there are no fixed costs such as management costs the solution is equivalent to the deterministic solution with

the timber value increasing at a constant exponential as established by Newman et al. (1985).<sup>8</sup> The following result then holds:

**Lemma 1** *When the timber price  $p$  follows the process (1a), the land value  $F(p)$  and the value of the stand at age  $a$ ,  $G(p, a)$ , are homogenous of degree one in  $p$ , and the optimal harvest age is independent of the timber price  $p$ , precisely,*

$$G(p, a) = ce^{\delta a} p \quad (11a)$$

$$F(p) = cp \quad (11b)$$

$$c = \frac{V(a_f)}{e^{\delta a_f} - 1} \quad (11c)$$

$$\frac{V_a(a_f)}{V(a_f)} = \frac{\delta}{1 - e^{-\delta a_f}} \quad (11d)$$

where  $a_f$  is the optimal harvest age.

**Proof.** See the appendix ■

Both land and forest values  $F(p)$  and  $G(p, a)$  depend linearly on the current price of the unique tree species. The optimal harvest age is constant from one harvest to the next, thus independent of price, and equals the Faustman (constant price) rotation if and only if the drift of the stochastic timber price process is zero. We call it the generalized Faustman age in this paper. It is also smaller than the Wicksellian one-rotation optimal harvest age given by (3c) as  $\frac{V_a(a)}{V(a)}$  is decreasing in  $a$ .

We now turn to a situation not previously considered in the literature. Two tree species  $P$  and  $P'$  are available. The DM has the option to harvest and to replant any combination of species, immediately after the harvest or after any delay, and to harvest again. The process goes on forever as establishing a new stand opens up the option to harvest, etc. As in the case of a single harvest, it can be shown that, to an expected value maximizer, any forest diversification strategy involving growing two species simultaneously would be dominated by one consisting in establishing one single

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<sup>8</sup>Saphores (2003) generalizes Faustman's formula to partial or total harvests in the case of a biomass whose stochastic growth is stock-dependent and represents the sole source of uncertainty. The optimal biomass at which harvest should occur is not a monotonic function of uncertainty, which implies that the relationship with Faustmann's rotation is ambiguous.



species if at all. Thus suppose that the problem starts with trees of either species established on the forest tract. The DM may choose (i) to wait and observe the stand growing up, or (ii) to harvest the stand and replant the same species immediately, or (iii) to harvest the stand and plant the alternative species, or finally (iv) to harvest the stand and wait before planting one or the other species. As before we assume that there is no fixed cost associated with harvesting and establishing a new crop, so that the role of the alternative species is highlighted.

Consider the last possibility. After harvesting, the forest DM may wait before establishing a new crop. Then the land remains bare,<sup>9</sup> allowing the DM to wait until timber prices evolve in such a way that it is easier to choose the right species. No gain can be achieved by using that strategy, though. Indeed suppose the DM decides to establish one species at some date  $t$  strictly posterior to the harvest. Had she planted that species immediately after harvest, she would then be better off at  $t$  because the trees would have grown already. Had she planted the other species, she could cut the trees and plant the preferred species at no cost. In that case she would either be as well off, if cutting the existing trees produced no income, or she would be better off if cutting the existing trees produced some income. This proves the following proposition, which implies that only the first three possible decisions outlined in the past paragraph need be considered.

**Proposition 3** *It is optimal to establish a new crop immediately after harvesting.*

This result contrasts with the one indicating that it is optimal, in the Wicksellian two-species single-harvest case, to delay planting when the projects are not clearly differentiated. Indeed, the decision to choose a species that turns out to be undesirable *ex post* has a lower opportunity cost in case of multiple rotations: trees that turn out to be less desirable after planting may be cut and replaced with the desirable species. In the single rotation case, the possibility of a more profitable harvest is lost once the wrong species is established.

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<sup>9</sup>Allowing for costs of keeping the land bare, such as weeding or protection against erosion, would only reinforce the result.

Given that the land is populated with one uniform-age species, consider now alternatives (i) – (iii) for harvest age and replacement choice. Let  $F(p, p')$  denote the value of the bare land, the value of the options to indefinitely plant and harvest the forest tract, choosing the appropriate species after each harvest. Let  $G(p, p', a)$  (respectively  $G'(p', p, a)$ ) denote the value of the forest (land and trees) when the forest tract is populated with trees of species  $P$  (respectively  $P'$ ) of age  $a$  and price  $p$  (resp.  $p'$ ) while the price for the other species is  $p'$  (resp.  $p$ ). Precisely,

$$G(p, p', a) = \max_s E e^{-rs} [p_s V(s+a) + F(p_s, p'_s)] \quad (12a)$$

$$G'(p', p, a) = \max_s E e^{-rs} [p'_s V'(s+a) + F(p_s, p'_s)] \quad (12b)$$

The following three lemmas give additional properties of the bare-land and forested-land functions that will help characterize the solution in terms of optimum harvest age and trigger price. The first lemma establishes the homogeneity of the functions.

**Lemma 2** *When  $p$  and  $p'$  follow the processes (1a) and (1b) respectively,  $F(p, p')$ ,  $G(p, p', a)$ , and  $G'(p', p, a)$  are homogenous of degree one in  $(p, p')$  and the optimal harvest age depends only on the current relative price of timber  $\theta = \frac{p'}{p}$ . Furthermore, the following reduced functions*

$$g(\theta, a) \equiv \frac{1}{p} G(p, p', a) \quad (13a)$$

$$g'(\theta, a) \equiv \frac{1}{p} G'(p', p, a) \quad (13b)$$

$$f(\theta) \equiv \frac{1}{p} F(p, p') \quad (13c)$$

*solve the following problems respectively*

$$g(\theta, a) = \max_s \left[ e^{-\delta s} V(a+s) + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right] \quad (14a)$$

$$g'(\theta, a) = \max_s \left[ e^{-\delta' s} \theta V'(a+s) + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right] \quad (14b)$$

$$f(\theta) = \begin{cases} f^-(\theta) \equiv \max_s \left[ e^{-\delta s} V(s) + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right], & \theta \leq \theta^* \\ f^+(\theta) \equiv \max_s \left[ e^{-\delta' s} \theta V'(s) + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right], & \theta \geq \theta^*. \end{cases} \quad (14c)$$

where  $\theta^*$  is the relative price below (above) which it is optimal to plant Species  $P$  (Species  $P'$ ) on a bare land.

**Proof.** xx vérifier; il me semble que la preuve en annexe est inutile. Il faut vérifier toutes les preuves

As  $g(\theta, a) \equiv \frac{1}{p}G(p, p', a) = \frac{1}{p} \max E e^{-rs} [p_s V(s+a) + F(p_s, p'_s)]$  then  $g(\theta, a) = \max_s \left[ V(s+a) \frac{e^{-rs} E p_s}{p} + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right]$ . Since  $E p_s = p e^{-\mu s}$ , it follows that  $g(\theta, a) = \max_s \left[ e^{-\delta s} V(s+a) + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right]$ . Note that  $\frac{p_s}{p}$  does not depend on  $P$  as it is a GBM that is equal to one at date zero. A similar proof gives (14b) and (14c), where the first (second) line of (14c) is obtained using the fact that the first harvest established on the bare land is Species  $P$  (Species  $P'$ ) when  $\theta \leq \theta^*$  ( $\geq \theta^*$ ). The existence of  $\theta^*$  is proven in Lemma 3. ■

As a consequence of Proposition 3, it is certain that the trees being harvested are replaced immediately. At relative prices below some switching value  $\theta^*$ , species  $P$  is planted if the land is bare; above  $\theta^*$ , it is optimal to plant species  $P$  on a bare land. It is certain that  $\theta^*$  exists as species  $P$  must be chosen when  $\theta$  tends to zero and species  $P$  must be chosen when  $\theta$  tends to infinity. This is formally proven in the next lemma.

**Lemma 3** *There exists a unique, strictly positive, value of  $\theta$ ,  $\theta^*$  such that, if  $\theta < \theta^*$  (respectively  $\theta > \theta^*$ ) and the land is bare, it is optimal to plant species  $P$  (respectively species  $P'$ ) immediately, while indifferently planting  $P$  or  $P'$  on bare land is optimal if  $\theta = \theta^*$ .*

**Proof.** The set of  $\theta > 0$  for which it is optimal to plant  $P$  is not empty as for  $\theta$  positive and sufficiently small it is optimal to plant  $P$ . This set is bounded as for  $\theta$  sufficiently high it is optimal to plant  $P'$ . Being not empty and bounded, the set of  $\theta > 0$  for which it is optimal to plant  $P$  has a finite maximum  $\theta^*$ . The unicity of  $\theta^*$  results from continuity. ■

Both the forested land and the bare land value functions increase when the price of either species increases. As a matter of fact, if the land is populated with one species

and the price of the other species increases while the price of the species in place does not change, the value of the forested land increases, because the exercise of the option to plant the other species becomes more likely. Furthermore, when relative prices are such that a switch to the alternative species might be a profitable decision, one should expect that the value functions should be higher than if that possibility to switch was not available. On the contrary, when relative prices are extreme, the land value should reflect the expected present value of the revenues associated with establishing and harvesting the same species forever,  $pc$  and  $p'c'$  for species  $P$  and  $P'$  respectively. Formally, for the reduced form of the land value, this gives:

**Lemma 4** *The reduced land value function is such that*

$$f(\theta) = g(\theta, 0) = g'(\theta, 0) \quad (15)$$

Furthermore,  $f(\theta)$  is increasing in  $\theta$  with  $\lim_{\theta \rightarrow 0} f(\theta) = c$ ,  $\lim_{\theta \rightarrow +\infty} f(\theta) = c'\theta$ , where  $c = \frac{V(a_f)}{e^{\delta a_f} - 1}$  and  $c' = \frac{V'(a'_f)}{e^{\delta' a'_f} - 1}$  and, for  $\theta > 0$ ,  $f(\theta) > c$  and  $f(\theta) > c'\theta$ .

**Proof.** The land value function  $F(p, p')$  must be strictly increasing in  $p$ ; then  $F_p(p, p') > 0$ . As  $F_p(p, p') = f_\theta(\theta)$  then  $f_\theta(\theta) > 0$  and therefore  $f(\theta)$  is strictly increasing in  $\theta$ . As the ratio of two GMB's,  $\theta$  is a GMG and zero is an absorbing value. When  $\theta$  tends to zero, that is when  $p'$  is close to zero while  $p$  is strictly positive with  $p' \ll p$ ,  $P'$  remains the preferred species for an indefinite time and the problem collapses to the single species case. Therefore, using (11b),  $\lim_{\theta \rightarrow \infty} pf(\theta) = cp$  or  $\lim_{\theta \rightarrow 0} f(\theta) = c$ . A similar proof shows that  $\lim_{\theta \rightarrow +\infty} f(\theta) = c'\theta$ . In the continuation region,  $G(p, p', a) = \max_s Ee^{-rs} [p_s V(a + s) + F(p_s, p'_s)]$ , i.e.  $G(p, p', 0) = F(p, p')$  or  $f(\theta) = g(\theta, 0)$  if  $(\theta, 0)$  belongs to the continuation region otherwise  $G(p, p', 0) = F(p, p') + pV(0)$  then  $f(\theta) = g(\theta, 0)$  as  $V(0) = 0$ . Similarly, one can show that  $f(\theta) = g'(\theta, 0)$ . Inequalities  $f(\theta) > c$  and  $f(\theta) > c'\theta$  follow from the fact that choosing species  $P$  or  $P'$  forever is feasible, yielding values of  $c$  and  $c'\theta$  respectively. The inequalities are strict because, for  $0 < \theta < \infty$ , there is a positive probability of switching species in such a way that the objective function is increased from either  $c$  or  $c'\theta$ . ■

5. CHOOSING BETWEEN ALTERNATIVE REPLACEMENTS REPEATEDLY: THE SOLUTION

We are interested in characterizing the continuation region when species  $P$  is in place, and the continuation region when species  $P'$  is in place. In either case the continuation region is the set of points  $(\theta, a)$  in  $\mathbb{R}^+ \times \mathbb{R}^+$  where it is optimum to wait rather than exercise the option of harvesting.<sup>10</sup> Suppose that species  $P$  is in place, a situation which is desirable at relatively low values of  $\theta$ . Clearly, as  $\delta > 0$ , there is an age above which it is desirable to cut the trees so that the continuation region, if it exists, is bounded above (along the  $a$  dimension). Let  $a^+(\theta)$  denote the upper boundary of the continuation region, the age above which it is optimum to harvest and below which it is desirable to allow the trees to grow further, given  $\theta$ . At high enough values of  $\theta$ , the alternative species  $P'$  becomes so attractive that there is no age at which it would be desirable to allow trees of species  $P$  to grow any further. At such high values of  $\theta$ , the continuation region of species  $P$  does not exist. Let  $\bar{\theta}$  denote the value of  $\theta$  above which the continuation region does not exist for any harvest age and below which the continuation region exists for some age. Thus  $\bar{\theta}$  delimits the right-hand end of the continuation region of Species  $P$ .

It can be shown by contradiction that the continuation region exists for any  $\theta < \bar{\theta}$  and that  $\bar{\theta}$  is strictly higher than  $\theta^*$ . Indeed, at  $\theta = \theta^*$  the decision maker is indifferent between stands of either species  $P$  or  $P'$  populated with trees of age zero, which respectively take periods of  $\underline{a}$  or  $\underline{a}'$  before producing any valuable timber. Then if the stand is populated with trees of species  $P$  whose age is strictly between 0 and  $\underline{a}$  while  $\theta = \theta^*$ , then it is strictly preferable to hold on to such a stand than to cut the trees, as cutting yields no revenue since  $a < \underline{a}$  and no benefit in terms of establishing the preferred species, since  $\theta = \theta^*$ . This proves that  $\theta^* < \bar{\theta}$ . This simple argument also implies that, for values of the relative price in  $[\theta^*, \bar{\theta}]$ , it is optimal to allow the trees to continue growing if they are older than some minimum age. Consequently, the continuation region has a lower

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<sup>10</sup>While  $\theta$  is the same variable in both continuation regions,  $a$  is specific to the species in place.

boundary  $a^-(\theta) > 0$  for  $\theta \in [\theta^*, \bar{\theta}]$ .

A similar analysis applies to the continuation region when species  $P'$  is in place. Remembering that  $\theta'^* \equiv \theta^*$ , it follows that  $\bar{\theta}' \leq \theta^*$  and that the continuation region for  $P'$  has upper and lower boundaries  $a'^+(\theta)$  and  $a'^-(\theta)$  on  $[\bar{\theta}', \theta^*]$ , and has only an upper boundary  $a'^+(\theta)$  on  $[\bar{\theta}', +\infty[$ . While the relative price  $\theta$  is the same variable in both cases, only considered at different values, one should note that the expressions corresponding to each region are not the mirror image of each other, although the analysis obeys a symmetric logic. For example, obviously,  $\bar{\theta}' \leq \theta^* \leq \bar{\theta}$  as species  $P'$  is desirable at high relative prices while the continuation region for  $P'$  does not exist below  $\bar{\theta}'$  by definition of  $\bar{\theta}'$ . Also, the relevant value functions (14a), (14b) and (14c) are not identical at values of  $\theta$  on either side of  $\theta^*$ , implying differences in the continuation regions.

Precisely, if species  $P$  is in place, then the upper boundary  $a^+(\theta)$  is the smallest value of  $a$  such that

$$\arg \max_s \left[ e^{-\delta s} V(a+s) + e^{-rs} E \frac{p_s}{p} f(\theta_s) \right] = 0 \quad (16)$$

The lower boundary  $a^-(\theta)$  is defined by the conditions

$$\text{For } \theta \in [\theta^*, \bar{\theta}], \quad \begin{aligned} g(\theta, a) &= V(a) + f(\theta), \text{ if } a = a^-(\theta) \\ g(\theta, a) &> V(a) + f(\theta), \text{ if } a^-(\theta) < a < a^+(\theta) \\ g(\theta, a) &< V(a) + f(\theta), \text{ if } a < a^-(\theta) \text{ or } a > a^+(\theta) \end{aligned} \quad (17)$$

For  $\theta \in [\theta^*, \bar{\theta}]$  and  $a \in ]a^-(\theta), a^+(\theta)[$ , the condition  $g(\theta, a) > V(a) + f(\theta)$  indicates that, if the land is populated with trees of species  $P$  and age  $a$ , it is preferable to allow them to reach maturity and harvest them when they reach age  $a^+(\theta)$ , rather than harvesting them immediately and planting species  $P'$  to obtain  $V(a)$  plus the expected land value  $f(\theta)$ . For  $\theta \in [\theta^*, \bar{\theta}]$  and  $a < a^-(\theta)$  or  $a > a^+(\theta)$ , the trees are too young to be allowed to reach maturity or are beyond maturity; condition  $g(\theta, a) < V(a) + f(\theta)$  then indicates that it is optimal to harvest immediately and switch to Species  $P'$ .

For  $\theta \in [0, \bar{\theta}]$ , if it is optimal to wait and harvest later on, the forest value function  $G(p, p', a)$  should satisfy Bellman's equation  $E(dG(p, p', a)) = rG(p, p', a)da$  which can be written as the following partial differential equation governing the reduced forest

value function  $g(\theta, a)$  in the continuation region (proof in the appendix):

$$\frac{\bar{\sigma}^2}{2}\theta^2 g_{\theta\theta} + \bar{\mu}\theta g_\theta - \delta g + g_a = 0 \quad (18)$$

Thus, on the upper boundary  $a^+(\theta)$  of the continuation region, the following value matching and smooth pasting conditions apply:

$$g(\theta, a^+(\theta)) = V(a^+(\theta)) + f(\theta) \quad (19a)$$

$$g_\theta(\theta, a^+(\theta)) = f_\theta(\theta) \quad (19b)$$

$$g_a(\theta, a^+(\theta)) = V_a(a^+(\theta)) \quad (19c)$$

The smooth pasting condition can be written in either of the two forms (19b) or (19c); given the value-matching condition these two forms are equivalent.<sup>11</sup> Clearly, no similar conditions apply to the lower boundary, as  $a^-(\theta)$  is not an optimum stopping unlike  $a^+(\theta)$ .

The following proposition enunciates properties of the upper and lower boundaries.

**Proposition 4** *Suppose that species  $P$  is currently in place. The upper boundary  $a^+(\theta)$  is continuous on  $[0, \bar{\theta}]$ , such that  $a^+(0) = a_f$ , and non increasing at low values of  $\theta$ ; on  $]\theta^*, \bar{\theta}[\blacksquare]$ ,  $a^+(\theta)$  is strictly decreasing. The lower boundary  $a^-(\theta)$  is continuous and strictly increasing on  $[\theta^*, \bar{\theta}]$ , with  $a^-(\theta^*) = 0$ . The upper and lower boundaries meet at  $(\bar{\theta}, \underline{a})$ .*

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<sup>11</sup>One can show that the reduced forest value function  $g'(\theta, a)$  should satisfy the same partial differential equation as  $g(\theta, a)$ , that is

$$\frac{\bar{\sigma}^2}{2}\theta^2 g'_{\theta\theta} + \bar{\mu}\theta g'_\theta - \delta g' + g'_a = 0$$

The boundary conditions are slightly different because the definition of the relative price  $\theta$  is maintained irrespective of the species in place. On the upper boundary  $a^{+'}(\theta)$  of the continuation region corresponding to  $g'$ , the following value matching and smooth pasting conditions apply:

$$g'(\theta, a^{+'}(\theta)) = \theta V'(a^{+'}(\theta)) + f(\theta)$$

$$g'_\theta(\theta, a^{+'}(\theta)) = V'(a^{+'}(\theta)) + f_\theta(\theta)$$

$$g'_a(\theta, a^{+'}(\theta)) = \theta V'_a(a^{+'}(\theta)).$$

**Proof.** When  $\theta$  tends to zero, the problem collapses to the single species case analyzed above, for which the optimum harvest age is  $a_f$  given by (10) i.e.  $a^+(0) = a_f$ .

When  $\theta$  is sufficiently close to zero, Equation (18) implies  $-\delta g + g_a \simeq 0$  or, using (19a) and (19c),  $\delta f(\theta) \simeq V_a(a^+(\theta)) - \delta V(a^+(\theta))$  where  $f(\theta)$  is non decreasing in  $\theta$  while  $V_a(a) - \delta V(a)$  strictly decreases in  $a$ . It follows that  $a^+(\theta)$  is non increasing on some interval that includes  $\theta = 0$ .

The continuity of  $a^+(\theta)$  follows from the continuity of the functions  $g$ ,  $f$ , and  $V$  that define it in expressions (19a), and (19b) or (19c). This argument also applies at  $\theta = \theta^*$  because functions  $g$  and  $f$  are continuous at  $\theta^*$  despite the change of definitions that they experience at that point according to (14c). The continuity of  $a^-(\theta)$  follows from the continuity of the functions that define it in (17).

Let us show that  $a^+(\theta)$  is strictly decreasing on  $]\theta^*, \bar{\theta}]$ . Assume that there exists a point  $A(\theta, a)$  on the boundary where  $a^+(\theta)$  is non decreasing while  $\theta > \theta^*$ . At  $A(\theta, a)$  the DM is indifferent between harvesting now and establishing species  $P'$  (the left-hand side of (20)), or waiting and harvesting later (the right-hand side of (20))<sup>12</sup>:

$$V(a) + \max_t \left[ e^{-\delta' t} \theta V'(t) + e^{-rt} E_\theta \left\{ \frac{p_t}{p} f(\theta_t) \right\} \right] = \max_s \left[ e^{-\delta s} V(a+s) + e^{-rs} E_\theta \left\{ \frac{p_s}{p} f(\theta_s) \right\} \right] \quad (20)$$

where  $E_\theta$  denotes expectation, conditional on  $\frac{p_s}{p}$  and  $\theta_s$  having initial values of 1 and  $\theta$  respectively. Let  $\varphi(s, y)$  denote  $E_\theta \left\{ \frac{p_s}{p} f(\theta_s) \right\}$  where  $y_s = \frac{p_s}{p} f(\theta_s)$  follows a generalized Brownian motion with  $y = f(\theta)$ .<sup>13</sup> Let a small period of time  $da > 0$  elapse, over which  $\theta$  becomes  $\theta + d\theta$  and  $y$  becomes  $y + dy$ . Take the case  $d\theta < 0$  (a similar argument holds when  $d\theta > 0$ ) and consider the point  $B(\theta + d\theta, a + da)$ , to the upper left of  $A$ . Since  $a^+(\theta)$  is assumed to be non decreasing, point  $B$  lies inside the immediate-harvest region. It follows that

$$V(a+da) + \max_t \left[ e^{-\delta'(t-da)} (\theta + d\theta) V'(t) + e^{-r(t-da)} \varphi(t-da, y+dy) \right] \geq \max_s \left[ e^{-\delta(s-da)} V(a+s) + e^{-r(s-da)} \varphi(s-da, y) \right]$$

When  $da \downarrow 0$ , appropriate expansions of both sides give:

<sup>12</sup>Both sides of the equation are normalized by  $p$ .

<sup>13</sup>According to the convention introduced earlier about time subscripts,  $\theta$ ,  $p$ ,  $y$  refer to current values, while  $\theta_s$ ,  $p_s$ ,  $y_s$  refer to the values of the same variables after a period  $s$  has elapsed.



$$\begin{aligned}
& V(a) + V_a(a)da + \max_t \left\{ (1 + \delta' da) e^{-\delta' t} (\theta + d\theta) V'(t) + (1 + rda) e^{-rt} [\varphi(t, y) + \varphi_y(t, y) E dy + \frac{1}{2} \varphi_{yy}(t, y) E dy^2] \right. \\
& \geq \max_s \left\{ (1 + \delta da) e^{-\delta s} V(a + s) + (1 + rda) e^{-rs} [\varphi(s, y) + \varphi_y(s, y) E dy + \frac{1}{2} \varphi_{yy}(s, y) E dy^2 - \varphi_s(s, y) da] \right.
\end{aligned}$$

Note that  $\varphi_y(s, y) E dy + \frac{1}{2} \varphi_{yy}(s, y) E dy^2 - \varphi_s(s, y) da = da [\mu_y \varphi_y(s, y) + \frac{1}{2} \sigma_y^2 \varphi_{yy}(s, y) - \varphi_t(s, y)]$

where  $\mu_y \varphi_y(s, y) + \frac{1}{2} \sigma_y^2 \varphi_{yy}(s, y) - \varphi_t(s, y)$  is the Kolmogorov backward differential equation associated with  $\varphi(t, y)$  where  $\mu_y$  and  $\sigma_y^2$  are respectively the drift and volatility of the generalized Brownian motion followed by  $y$ . As such this expression is equal to zero (see for instance Karlin et al. (1981), page 214). Therefore, after neglecting terms of order of magnitude smaller than or equal to  $da$  under the assumption that  $da \downarrow 0$ , we obtain,  $V(a) + \max_t [e^{-\delta' t} (\theta + d\theta) V'(t) + e^{-rt} \varphi(t, y)] \geq \max_s [e^{-\delta s} V(a + s) + e^{-rs} \varphi(s, y)]$  which contradicts Equation (20) as  $d\theta$  was initially assumed to be  $< 0$ . Therefore  $a^+(\theta)$  is strictly decreasing on  $]\theta^*, \bar{\theta}]$ .

Similarly, let us show that the lower boundary  $a^-(\theta)$  is strictly increasing on  $[\bar{\theta}', \theta^*]$ . At a point  $C(\theta, a)$  on  $a^-(\theta)$ , with  $a < \underline{a}$  and  $\bar{\theta}' < \theta < \theta^*$ , assume that  $a^-(\theta)$  is non increasing. The DM is indifferent between harvesting immediately thus earning the land value (the harvest is worth zero as  $a < \underline{a}$ ) given by the left-hand term of the following equation, or harvesting later at age  $a + s > \underline{a}$ , in which case she earns the right-hand of the same equation:

$$\max_t \left[ e^{-\delta' t} \theta V'(t) + e^{-rt} E \frac{P_t}{p} f(\theta_t) \right] = \max_s \left[ e^{-\delta s} V(a + s) + e^{-rs} E \frac{P_s}{p} f(\theta_s) \right] \quad (21)$$

After a small time interval  $da > 0$ ,  $\theta$  becomes  $\theta + d\theta$ . Assuming that  $d\theta > 0$  and given that  $a^-(\theta)$  is non increasing, the pair  $C(\theta, a)$  moves to  $D(\theta + d\theta, a + da)$  in the continuation region. Proceeding as in the previous proof, we obtain  $\max_t [e^{-\delta' t} \theta V'(t) + e^{-\delta' t} d\theta V'(t) + e^{-\delta t} E \frac{P_t}{p} f(\theta_t)] \geq \max_s [e^{-\delta s} V(a + s) + e^{-\delta s} E \frac{P_s}{p} f(\theta_s)]$  which cannot hold along with Equation (21) as  $d\theta > 0$ . This implies that the boundary  $a^-(\theta)$  is strictly increasing on  $]\theta^*, \bar{\theta}]$ .

At  $(\theta^*, 0)$  the DM is indifferent between planting species  $P$  or switching to species  $P'$ ; thus  $(\theta^*, 0)$  belongs to  $a^-(\theta)$ . Also, by definition,  $a^-(\theta) \leq \underline{a}$ ; in particulier  $a^-(\bar{\theta}) \leq \underline{a}$ .

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<sup>14</sup>Strictly, the solution of the maximization problems on both sides of the inequality are affected by the expansion of the objective functions. However the derivation is valid whether or not the optimum timing of the harvest is forced to be maintained while the functions are expanded. We avoid introducing further notation by implicitly ignoring this issue as it has no bearing on the result.

Let us show by contradiction that  $a^+(\bar{\theta}) = \underline{a}$ . Suppose that  $a^+(\bar{\theta}) = \underline{a} + \varepsilon$ ,  $\varepsilon > 0$ . Then  $V(\underline{a} + \varepsilon) + f(\bar{\theta}) > V(\underline{a}) + f(\bar{\theta}) \geq V(a^-(\bar{\theta})) + f(\bar{\theta})$  where the left-hand side applies the definition (16) of the upper boundary while the right-hand side results from the condition that  $a^-(\bar{\theta}) \leq \underline{a}$ . But then, for some pair  $(\varepsilon'', \bar{\theta}'')$  such that  $0 < \varepsilon'' < \varepsilon$  and  $\bar{\theta}'' > \bar{\theta}$ , it is also true that  $V(\underline{a} + \varepsilon'') + f(\bar{\theta}'') > V(a^-(\bar{\theta}'')) + f(\bar{\theta}'')$  so that  $\bar{\theta}$  is not the maximum value of  $\theta$  at which the continuation region exists, contradicting its definition. This contradiction can be avoided only if  $\varepsilon = 0$ , implying that  $a^+(\bar{\theta}) = \underline{a}$ . Now suppose  $a^-(\bar{\theta}) < \underline{a}$ ; then by (14a)  $g(\bar{\theta}, \underline{a}) > V(a^-(\bar{\theta})) + f(\bar{\theta}) = f(\bar{\theta})$ . However,  $g(\bar{\theta}, a^-(\bar{\theta})) = f(\bar{\theta})$  by (17) since  $V(a) = 0 \forall a < \underline{a}$ , a contradiction. Hence  $a^-(\bar{\theta}) = \underline{a}$ . We conclude that the upper and lower boundaries meet at  $(\bar{\theta}, \underline{a})$ . ■

We are now going to argue and show that, when the value of  $\theta$  is not far from and below  $\theta^*$  so that a switch to Species  $P'$  at the next harvest may be forthcoming, the harvest age on the boundary is increasing in  $\theta$ .

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When  $\theta$  is approaching  $\theta^*$  from below, the decision to harvest  $\theta$  may have two alternative implications: if the trees are cut immediately, since  $\theta < \theta^*$ , Species  $P$  must be chosen after the harvest. Alternatively, allowing the trees to grow to some higher age may allow  $\theta$  to overtake  $\theta^*$  before the harvest so that Species  $P'$  would be established after the harvest. Thus the decision to wait rather than harvest opens the possibility of a switch to the species not currently in place, and the closer  $\theta$  is to  $\theta^*$ , the higher the probability that waiting will lead to a switch at the next harvest. The slope of the upper boundary reflects the desirability of keeping the option to switch at the coming harvest open; a positive slope means that the trees are harvested at an increasingly high age, the closer  $\theta$  is to  $\theta^*$ , in an attempt to keep the option of switching open.

Consider in Figure 1 a pair  $(a^+(\theta_1), \theta_1)$  on the upper boundary  $a^+(\theta)$ , with  $\theta_1 < \theta^*$ .

On the boundary the value from immediate harvest is identical to the expected value from delaying. If the boundary is downward sloping in  $\theta$ , delaying the harvest by any time interval  $dt$  cannot result in an eventual switch because, since  $da > 0$ , any increase in  $\theta$  would cause the  $(a, \theta)$  pair to cross the boundary, implying reestablishing the same species. On the contrary, if the boundary is upward sloping as depicted, delaying the harvest may allow the  $(a, \theta)$  pair to move to the right and up before crossing the boundary again. The harvest (and switch) then would happen at some value  $\theta_2$  such that<sup>15</sup>

$$a^+(\theta_1) < a^+(\theta_2), \theta_1 < \theta^* < \theta_2.$$

This shows that if the boundary is upwardsloping in  $\theta$  for some  $\theta_1 < \theta^*$  and harvesting is optimal at age  $a^+(\theta_1)$ , then delaying harvest leads to harvesting at a higher age and a higher level of  $\theta$ , with a positive probability of switch. *Vice versa*, if it is optimal to harvest immediately while equally optimal to keep the prospect of switching open by delaying harvest, then the slope of the boundary when  $\theta$  approaches  $\theta^*$  from below is positive as represented in Figure 1. It should be noted that that delaying harvest from Point A in Figure 1 may result in a harvest at Point B with a switch to Species  $P'$ , but may also result in a harvest at Point A' with no switch. A positive slope of the harvest boundary thus allows but does not guarantee a switch; a non positive slope rules out any switch.

Let us show that, when  $\theta$  approaches  $\theta^*$  from below and it is time to harvest, a good alternative strategy is to wait in the hope that the delay will lead to a species switch. The best way to understand that is to start by considering the certainty case. In that case  $\theta$  can approach  $\theta^*$  from below only if  $\frac{\mu'}{\mu}$ , the drift of  $\theta$ , is strictly positive....

The monotonicity properties of  $a^+(\theta)$  and  $a^-(\theta)$  stated in Proposition 4 are independent of the parameters of the price processes or the species growth functions. Therefore, by symmetry, if species  $P'$  is in place then  $a'^+(\theta)$  is strictly increasing while the lower boundary  $a'^-(\theta)$  is strictly decreasing on  $[\bar{\theta}', \theta^*]$  as depicted on Figure 1.

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<sup>15</sup>This possibility does rule out another possibility, that the pair  $(a, \theta)$  hits the barrier again at some value of  $\theta$  short of  $\theta^*$ .

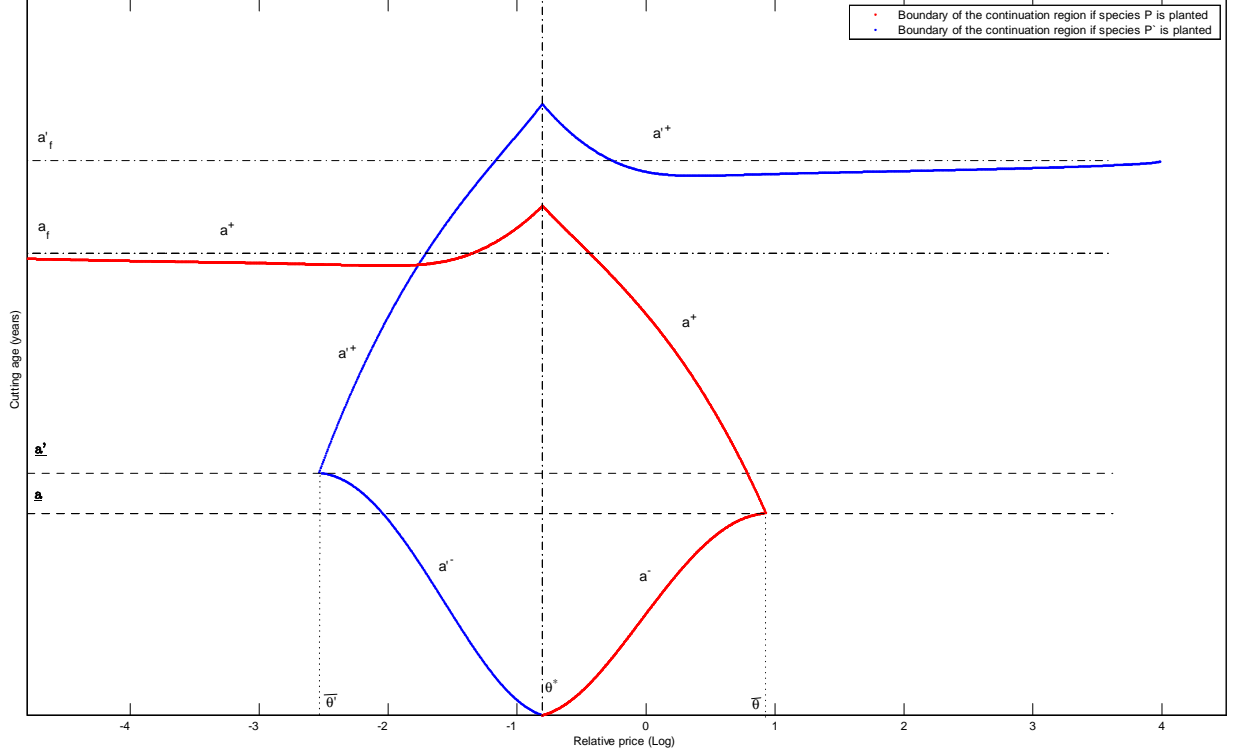


Figure 1: Boundaries of the continuation region according to the species in place

Compared with Faustman’s rule, including its versions with price uncertainty, the presence of an alternative tree species modifies the harvesting and planting decision substantially.

### 5.1 Numerical resolution

In order to conclude the description of the solution of the model, in particular in order to draw the boundary of the continuation region, it is necessary to use numerical methods. The reduced forest value function  $g(\theta, a)$  must satisfy the partial differential equation (18) and the value-matching and smooth pasting conditions (Equations (19a), (19b), and (19c)). The partial differential equation governing  $g(\theta, a)$  can be simplified by performing the change of variable  $x = \log(\theta)$ . Let  $h(x, a) = g(\theta, a)$  and  $l(x) = f(\theta)$ , then the partial differential equation governing  $h(x, a)$  is

$$\frac{\bar{\sigma}^2}{2}h_{xx} + \left(\bar{\mu} - \frac{\bar{\sigma}^2}{2}\right)h_x - \delta h + h_a = 0 \quad (22)$$

The optimal stopping problem of valuing the forest value when species  $P$  is planted is similar to the problem of valuing an American-type option with free boundary. Because the free boundary location is not known in advance, the value-matching and smooth pasting conditions cannot be of immediate help. These conditions can be used to localize the free harvest boundary once the stand value is determined. We do so iteratively as explained later in this section. To compute the forest value function  $h(x, a)$ , it is helpful to specify the corresponding optimal stopping problem as a linear complementarity one (Zvan et al, 1998). Let  $\mathcal{L}$  be the linear operator defined as

$$\mathcal{L}.h = \frac{\bar{\sigma}^2}{2}h_{xx} + \left(\bar{\mu} - \frac{\bar{\sigma}^2}{2}\right)h_x - \delta h + h_a$$

Then, the linear complementarity problem is

$$\begin{aligned} \mathcal{L}.h(x, a) &\geq 0 \\ h(x, a) - (l(x) + V(a)) &\geq 0 \\ \mathcal{L}.h(x, a) [h(x, a) - (l(x) + V(a))] &= 0 \end{aligned}$$

Note that this formulation does not imply any explicit use of the free harvest boundary  $a(\theta)$ . It shows that the value function  $h(x, a)$  can be considered as the value of an American option with expiry date equal to infinity, an underlying asset which is a geometric diffusion process with drift  $\bar{\mu}$  and volatility  $\bar{\sigma}^2$ , and a discount factor equal to  $\delta$ . Indeed, in the continuation region where it is optimal to continue holding the option to harvest, the required return  $\delta h$  is equal to the actual return or equivalently  $\mathcal{L}.h(x, a) = 0$  and the option value is higher than the payoff, that is  $h(x, a) - (l(x) + V(a)) > 0$ . Consequently, it is not yet optimal to exercise. In the stopping region, it is no more optimal to continue holding the option to harvest but it is optimal to harvest immediately because the required return  $\delta h$  is less than the actual return or equivalently  $\mathcal{L}.h(x, a) > 0$  and the option value must equal the payoff that is  $h(x, a) - (l(x) + V(a)) = 0$ . The

free harvest boundary is just where the decision maker is indifferent between harvesting immediately or continuing to hold this option, that is when  $\mathcal{L}.h(x, a) = 0$  and  $h(x, a) - (l(x) + V(a)) = 0$ .

When valuing an American option, the use of the complementarity formulation is straightforward. The option value is then computed numerically by performing a discretization of the linear complementarity problem. The main difference between valuing an American option and the forest value is that the payoff in the former case is known as a function of the underlying asset when exercising the option whereas it should be endogenously determined in the latter as it is the sum of the timber crop value and the land value  $l(x) = h(x, 0)$ . For this reason, valuing the forest value function  $h(x, a)$  as a complementarity problem as specified above is seemingly not possible. In order to overcome this limitation, we need to consider simultaneously the second problem consisting in valuing the forest value function  $g'(\theta, a) = h'(x, a)$  as follows:

$$\mathcal{L}.h(x, a) \geq 0 \tag{23a}$$

$$h(x, a) - (l(x) + V(a)) \geq 0 \tag{23b}$$

$$\mathcal{L}.h(x, a) [h(x, a) - (l(x) + V(a))] = 0 \tag{23c}$$

$$\mathcal{L}.h'(x, a) \geq 0 \tag{24a}$$

$$h'(x, a) - (l(x) + e^x V(a)) \geq 0 \tag{24b}$$

$$\mathcal{L}.h'(x, a) [h'(x, a) - (l(x) + e^x V(a))] = 0 \tag{24c}$$

$$l(x) = h(x, 0) = h'(x, 0) \tag{25}$$

Equations (23a)–(24c) represent the complementarity problem respectively for stand  $P$  and stand  $P$  whereas equation (25) says that each forest value at age zero must equal the land value by Proposition 4. As specified by equations (23a) – (24c), and equation (25), the problem of valuing simultaneously both stands can then be solved numerically.

The value-matching and smooth-pasting conditions will be used to localize the harvest boundary. The forest value functions  $h(x, a)$  and  $h'(x, a)$  are computed iteratively as specified by the following pseudo-code. First, (i) assume that after the  $n - 1^{th}$  iteration, the land value function is  $l^{(n)}$ ; then (ii) compute forest value function  $h^{(n)}$  as solution to (23a) – (23c) and forest value function  $h'^{(n)}$  as solution to (24a) – (24c); (iii) deduce the new land value function as  $l^{(n+1)}(x) = \max(h^{(n)}(x, 0), h'^{(n)}(x, 0))$ ; (iv) continue iteratively until convergence is reached when  $h(x, 0)$  and  $h'(x, 0)$  are approximately equal.

The initial value ascribed to the land value can be arbitrarily chosen. We find that the algorithm converges when the land value has initially any positive values. However, the convergence is generally faster when the initial land value function is equal to  $\max(c, c'\theta)$  or  $\max(c, c'e^x)$ . Indeed, this is the minimal land value when both tree species  $P$  and  $P'$  are available.

The numerical discretization to compute  $h^{(n)}$  and  $h'^{(n)}$  at step (ii) is based on a fully implicit finite difference method. With respect to a fully explicit finite difference method, the implicit method is unconditionally stable and more robust (Brennan and Schwartz 1978). Other numerical methods to solve option valuation problems are discussed in Wilmott et al (1993). We use the penalty method (Zvan et al, 1998) to solve the linear complementarity problem as in Insley (2002).

Figures 2 and 3 illustrate the harvest age boundary with respectively low and high uncertainty in the case where  $\mu < \mu'$ . The relative price threshold  $\theta^*$  for which the decision maker is indifferent between planting any of the two species on a bare land (the switching threshold) corresponds to the relative price for which it is optimal to cut the currently planted species at age zero. It is the same relative price at which the replanting harvest boundary meets the switching boundary.

Figure 4 shows the land value function in the case where  $\mu < \mu'$ . The land has a greater value under uncertainty than under certainty and it is more valuable when the uncertainty level, measured by  $\bar{\sigma}^2$ , increases. Unlike the case under certainty, the land value function has a continuous slope around the relative price switching threshold under uncertainty to prevent any arbitrage. When the relative price tends respectively to zero

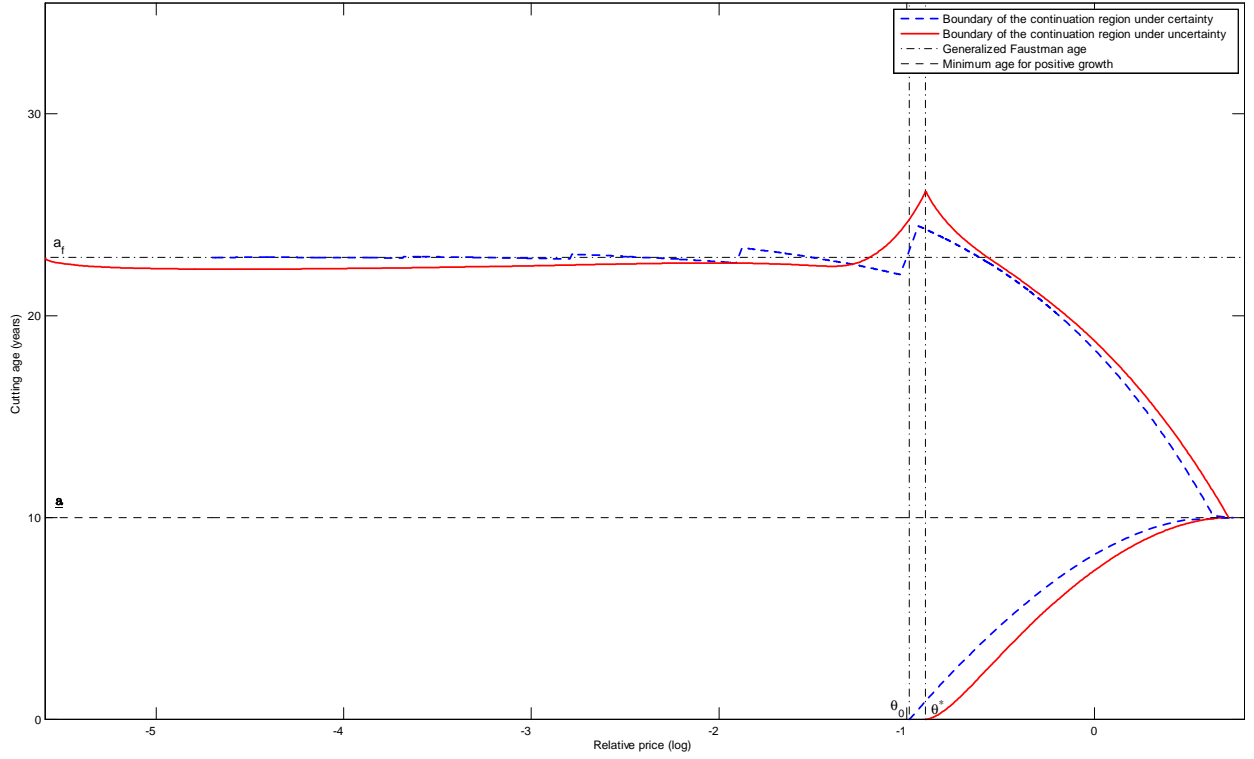


Figure 2: Boundary of the continuation region under low uncertainty ( $\mu < \mu'$ )

or to infinity, that is when respectively species  $P$  or  $P$  is worthless, the reduced land value functions tend toward their single species level  $c$  or  $c'\theta$  respectively.

## 6. CONCLUSION

We have examined the decision to undertake projects that differ in that they generate different future income flows. The focus has been extensions of the conventional forestry economics model. In our extended forestry model, two alternative species may be planted, so that outputs as well as timber prices differ across species. This entails more sophisticated planting and harvesting decisions than had been considered before.

When choosing between two alternative species for just one rotation, the decision maker plants immediately if the price of one species is sufficiently high compared to the price of the other species. However, the decision maker prefers to wait if both prices



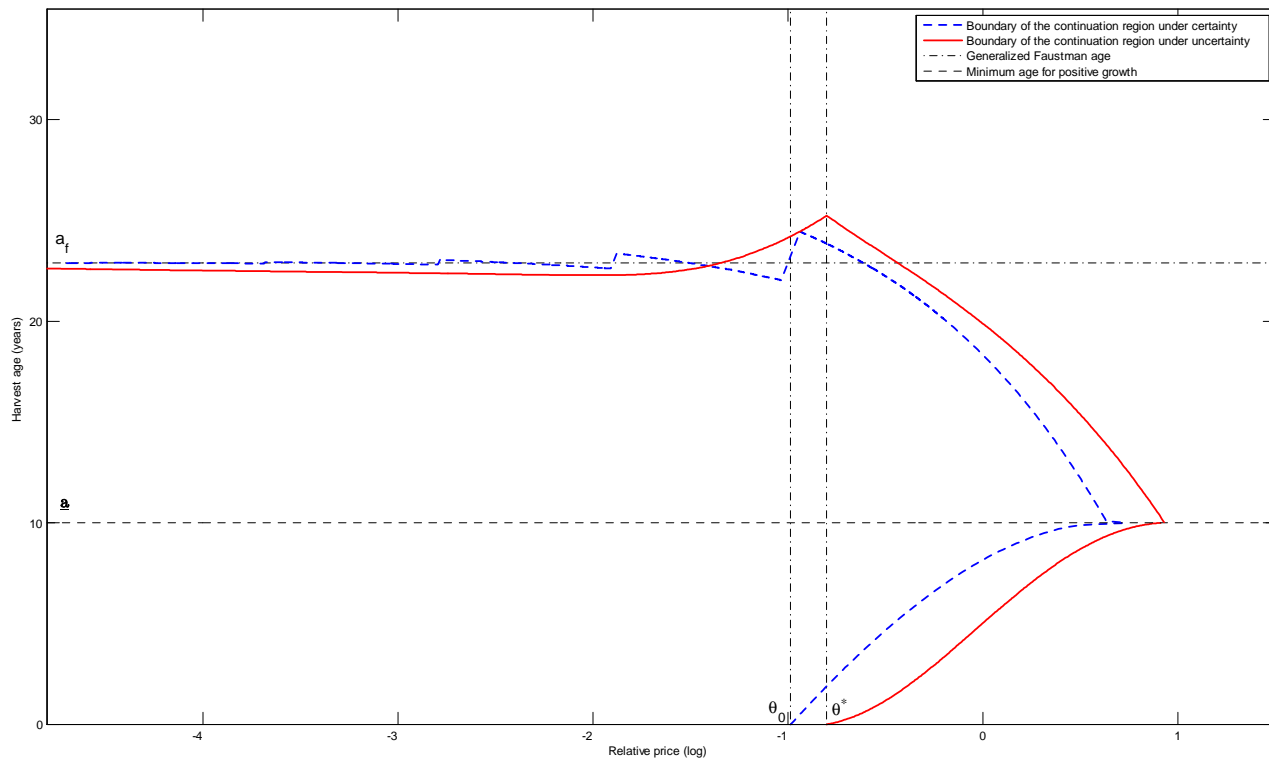


Figure 3: Boundary of the continuation region under high uncertainty ( $\mu < \mu'$ )

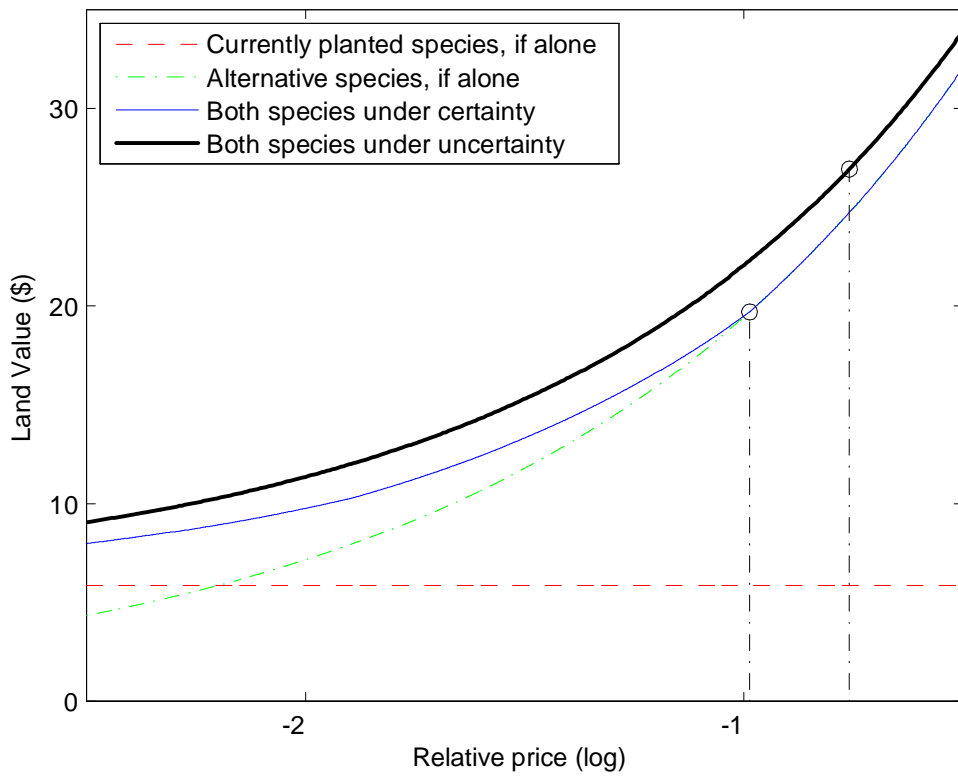


Figure 4: Land value as function of the relative price ( $\mu < \mu'$ )

are sufficiently close. This is so even while the decision does not involve any direct cost. Indeed, even in the absence of explicit investment costs, there is the potential cost of establishing the wrong species. This mistake is irreversible because the project involves one harvest only. Consequently, the decision maker prefers waiting until the prices of the alternative species are sufficiently far away from each other to make the probability of a price reversal acceptably low.

We then considered situations where the decision maker has the opportunity to exploit the forest land for an indefinite number of rotations. Here the decision maker must decide at what age the current stand should be harvested and whether the same species, or the alternative species, should be replanted, immediately or after some delay.

We showed that, absent any planting costs, the DM immediately establishes the species whose relative price exceeds some threshold; otherwise the alternative species is selected. This rule differs from the one rotation case because introducing many rotations has the effect of diluting the irreversibility effect present in the one rotation case. The decision maker plants immediately because it is possible to switch to the alternative species whenever desirable. This is not an easy decision, however, as trees may in that case have to be harvested while they have little or no value. We have shown that, although waiting without planting may help choose the best species, nothing can be gained and some loss may be made, by using that strategy.

We have characterized the value functions and the optimal management strategy in this stochastic repeated rotations context, although they were not provided in explicit form. In the space of relative species prices and wood stand ages, we characterized the set of points where the DM is willing to wait or to exercise the option of harvesting. The exercise frontier divides itself into a zone where the same species is reestablished immediately, and, at higher relative prices a zone where the other species is chosen. The relative price that separates these two zones is independent of stand age. At still higher relative prices, the continuation zone and exercise frontier disappear altogether: a stand of the "wrong" species should then be harvested immediately.

Land value is higher than when one species only is available. It converges to the

limiting cases of one species, or the other, when their relative price tends to zero or to infinity. When both species are available, the optimum cutting age is non monotonous but oscillates around Faustman's' age when the relative price is below some threshold value signalling the necessity to switch to the alternative species. This is because the decision maker would rather wait than plant the wrong species and adjust the number of times the same species has to be planted. The land value increases with the uncertainty related to timber prices. It is an increasing function of the relative price with a continuous slope even around the switching threshold to prevent any arbitrage as long as the uncertainty is present. On the contrary, under certainty, the slope of the land value function generally changes at the switching threshold. For relative prices higher than the switching threshold, the optimal harvesting age decreases until it hits the minimum age at which timber volumes become positive. Similar results are established explicitly in the certainty case.

## 7. APPENDIX

### a) Risk neutrality case:

Consider a risk neutral decision maker who chooses the proportion  $\lambda$  of a bare piece of land to be devoted to either one of the species while simultaneously choosing the planting date. The rest of the land  $1 - \lambda$  is devoted to the other species and planted simultaneously or later. Harvests take place at dates to be chosen optimally for each species. Assuming without loss of generality that species  $P$  is planted at  $T$  and species  $P$  is established at  $T'$ , the value of the project is then

$$F(p, p') = E \max_{T, T', \lambda \in [0, 1]} \left\{ \lambda e^{-rT} \max_s E_T \left( e^{-rs} V(s) p_{T+s} \right) + (1 - \lambda) e^{-rT'} \max_{s'} E_{T'} \left( e^{-rs'} V'(s') p'_{T'+s'} \right) \right\} \quad (26)$$

where  $E$  is the expected value operator conditional on current information, and  $E_T$  (respectively  $E_{T'}$ ) is the expected value operator conditional on information at  $T$  (respectively  $T'$ ). Since  $\lambda$  and  $1 - \lambda$  do not affect the choices of  $s$  or  $s'$  in problem (26), and since they enter the objective function multiplicatively, they do not affect the choices of  $T$  or  $T'$ . Consequently  $\lambda$  can be determined given the optimal rules for choosing  $T$  and  $T'$ :

$$F(p, p') = E \max_{\lambda \in [0, 1]} \left\{ \lambda \max_T E \left( e^{-rT} \max_s E_T e^{-rs} V(s) p_{T+s} \right) + (1 - \lambda) \max_{T'} E \left( e^{-rT'} \max_{s'} E_{T'} e^{-rs'} V'(s') p'_{T'+s'} \right) \right\}$$

The above problem is linear in  $\lambda$ . Consequently the solution is either  $\lambda = 1$  or  $\lambda = 0$ . This establishes that diversification is not optimum.

### Proof of Proposition 1 XX A vérifier encore

The only elements not established in the text are the price thresholds and the constants in Formula (7). For a current price level  $p$ , sufficiently high relative to  $p'$ , that is for  $\theta$  smaller than a critical threshold  $\theta^*$ , it is optimal to invest immediately in  $P$ ; then, by (5),  $F(p, p') = bp$  or  $f(\theta) = b$ . On the boundary  $\theta = \theta^*$ , the value-matching and smooth-pasting conditions (Dixit, 1993) imply  $f(\theta^*) = b$  and  $f_\theta(\theta^*) = 0$ . Similarly, at levels  $p'$  sufficiently high relative to  $p$ , that is for  $\theta$  bigger than a critical threshold  $\theta'^*$ , it is optimal to invest immediately in  $P$ ; then  $F(p, p') = b'p'$  or  $f(\theta) = b'\theta$ , and the value-matching and smooth-pasting conditions are respectively  $f(\theta'^*) = b'\theta'^*$  and  $f_\theta(\theta'^*) = b'$ . Consider the following equations to determine  $b_1$ ,  $b_2$ ,  $\theta^*$  and  $\theta'^*$ :

$$b_1 \theta^{*\beta_1} + b_2 \theta^{*\beta_2} = b \quad (27a)$$

$$\beta_1 b_1 \theta^{*\beta_1 - 1} + \beta_2 b_2 \theta^{*\beta_2 - 1} = 0 \quad (27b)$$

$$b_1 \theta'^{\beta_1} + b_2 \theta'^{\beta_2} = b' \theta'^* \quad (27c)$$

$$\beta_1 b_1 \theta'^{\beta_1 - 1} + \beta_2 b_2 \theta'^{\beta_2 - 1} = b' \quad (27d)$$

Consider the first two equations 27a and 27b, then:

$$\begin{aligned}\beta_1 b_1 \theta^{*\beta_1} + \beta_1 b_2 \theta^{*\beta_2} &= b\beta_1 \\ \beta_1 b_1 \theta^{*\beta_1} + \beta_2 b_2 \theta^{*\beta_2} &= 0\end{aligned}$$

$$\text{then } (\beta_1 - \beta_2) b_2 \theta^{*\beta_2} = b\beta_1 \text{ or } \theta^* = \left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{\beta_2^{-1}} b_2^{-\beta_2^{-1}}$$

The same first two equations imply also:

$$\begin{aligned}\beta_2 b_1 \theta^{*\beta_1} + \beta_2 b_2 \theta^{*\beta_2} &= b\beta_2 \\ \beta_1 b_1 \theta^{*\beta_1} + \beta_2 b_2 \theta^{*\beta_2} &= 0\end{aligned}$$

$$\text{then } (\beta_1 - \beta_2) b_1 \theta^{*\beta_1} = -b\beta_2 \text{ or } \theta^* = \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_1^{-1}} b_1^{-\beta_1^{-1}}$$

$$\text{As } \theta^* = \left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{\beta_2^{-1}} b_2^{-\beta_2^{-1}} = \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_1^{-1}} b_1^{-\beta_1^{-1}} \text{ then } \left[ \left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{\beta_2^{-1}} b_2^{-\beta_2^{-1}} \right]^{\beta_1 \beta_2} = \left[ \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_1^{-1}} b_1^{-\beta_1^{-1}} \right]^{\beta_1 \beta_2}$$

and

$$\left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{\beta_1} b_1^{\beta_2} = \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2} b_2^{\beta_1} \quad (28)$$

Then consider the last two equations 27c and 27d; one can proceed similarly to obtain

$$(\beta_1 - \beta_2) b_2 \theta^{*\beta_2} = b' \theta^{*\beta_1} (\beta_1 - 1)$$

$$(\beta_1 - \beta_2) b_1 \theta^{*\beta_1} = b' \theta^{*\beta_2} (1 - \beta_2)$$

and by eliminating  $\theta^{*\beta_1}$

$$\left( \frac{b'(\beta_1 - 1)}{\beta_1 - \beta_2} \right)^{\beta_1 - 1} b_1^{\beta_2 - 1} = \left( \frac{b'(1 - \beta_2)}{\beta_1 - \beta_2} \right)^{\beta_2 - 1} b_2^{\beta_1 - 1} \quad (29)$$

Substituting for  $b_1$  using equations 28 and 29:

$$\left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{\beta_1(\beta_1 - 1)} b_1^{\beta_2(\beta_1 - 1)} = \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_1 - 1)} b_2^{\beta_1(\beta_1 - 1)}$$

$$\left( \frac{b'(\beta_1 - 1)}{\beta_1 - \beta_2} \right)^{\beta_1(\beta_1 - 1)} b_1^{\beta_1(\beta_2 - 1)} = \left( \frac{b'(1 - \beta_2)}{\beta_1 - \beta_2} \right)^{\beta_1(\beta_2 - 1)} b_2^{\beta_1(\beta_1 - 1)}$$

so

$$b_1^{\beta_1 - \beta_2} = \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_1 - 1)} \left( \frac{b'(1 - \beta_2)}{\beta_1 - \beta_2} \right)^{-\beta_1(\beta_2 - 1)} \left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{-\beta_1(\beta_1 - 1)} \left( \frac{b'(\beta_1 - 1)}{\beta_1 - \beta_2} \right)^{\beta_1(\beta_1 - 1)}$$

$$b_1^{\beta_1 - \beta_2} = \left( \frac{-\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_1 - 1)} \left( \frac{1 - \beta_2}{\beta_1 - \beta_2} \right)^{-\beta_1(\beta_2 - 1)} \left( \frac{\beta_1}{\beta_1 - \beta_2} \right)^{-\beta_1(\beta_1 - 1)} \left( \frac{\beta_1 - 1}{\beta_1 - \beta_2} \right)^{\beta_1(\beta_1 - 1)} b^{\beta_2(\beta_1 - 1) - \beta_1(\beta_1 - 1)} b'^{-\beta_1(\beta_2 - 1) + \beta_1(\beta_1 - 1)}$$

$$b_1^{\beta_1 - \beta_2} = \left( \frac{-\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_1 - 1)} \left( \frac{1 - \beta_2}{\beta_1 - \beta_2} \right)^{\beta_1 - \beta_2} \left( \frac{1 - \beta_2}{\beta_1 - \beta_2} \right)^{-\beta_1(\beta_2 - 1) - (\beta_1 - \beta_2)} \left( \frac{\beta_1}{\beta_1 - 1} \right)^{-\beta_1(\beta_1 - 1)} b^{-(\beta_1 - 1)(\beta_1 - \beta_2)} b'^{\beta_1(\beta_1 - \beta_2)}$$

$$b_1^{\beta_1 - \beta_2} = \left( \frac{1 - \beta_2}{\beta_1 - \beta_2} \right)^{\beta_1 - \beta_2} \left( \frac{-\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_1 - 1)} \left( \frac{1 - \beta_2}{\beta_1 - \beta_2} \right)^{-\beta_2(\beta_1 - 1)} \left( \frac{\beta_1}{\beta_1 - 1} \right)^{-\beta_1(\beta_1 - 1)} b^{-(\beta_1 - 1)(\beta_1 - \beta_2)} b'^{\beta_1(\beta_1 - \beta_2)}$$

$$b_1^{\beta_1 - \beta_2} = \left( \frac{1 - \beta_2}{\beta_1 - \beta_2} \right)^{\beta_1 - \beta_2} \left( \frac{\beta_2}{\beta_2 - 1} \right)^{\beta_2(\beta_1 - 1)} \left( \frac{\beta_1}{\beta_1 - 1} \right)^{-\beta_1(\beta_1 - 1)} b^{-(\beta_1 - 1)(\beta_1 - \beta_2)} b'^{\beta_1(\beta_1 - \beta_2)}$$

$$b_1 = \frac{1 - \beta_2}{\beta_1 - \beta_2} \left( \frac{\beta_1 - 1}{\beta_1} \right)^{\beta_1(\beta_1 - 1) / (\beta_1 - \beta_2)} \left( \frac{\beta_2 - 1}{\beta_2} \right)^{-\beta_2(\beta_1 - 1) / (\beta_1 - \beta_2)} \frac{b'^{\beta_1}}{b^{\beta_1 - 1}}$$

Substituting for  $b_2$  using equations 28 and 29:

$$\left( \frac{b\beta_1}{\beta_1 - \beta_2} \right)^{\beta_1(\beta_2 - 1)} b_1^{\beta_2(\beta_2 - 1)} = \left( \frac{-b\beta_2}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_2 - 1)} b_2^{\beta_1(\beta_2 - 1)}$$

$$\left( \frac{b'(\beta_1 - 1)}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_1 - 1)} b_1^{\beta_2(\beta_2 - 1)} = \left( \frac{b'(1 - \beta_2)}{\beta_1 - \beta_2} \right)^{\beta_2(\beta_2 - 1)} b_2^{\beta_2(\beta_1 - 1)}$$

Then

$$\begin{aligned}
b_2^{\beta_1-\beta_2} &= \left(\frac{b'(\beta_1-1)}{\beta_1-\beta_2}\right)^{\beta_2(\beta_1-1)} \left(\frac{b\beta_1}{\beta_1-\beta_2}\right)^{-\beta_1(\beta_2-1)} \left(\frac{b'(1-\beta_2)}{\beta_1-\beta_2}\right)^{-\beta_2(\beta_2-1)} \left(\frac{-b\beta_2}{\beta_1-\beta_2}\right)^{\beta_2(\beta_2-1)} \\
b_2^{\beta_1-\beta_2} &= \left(\frac{\beta_1-1}{\beta_1-\beta_2}\right)^{\beta_1-\beta_2} \left(\frac{\beta_1-1}{\beta_1-\beta_2}\right)^{\beta_2(\beta_1-1)-(\beta_1-\beta_2)} \left(\frac{\beta_1}{\beta_1-\beta_2}\right)^{-\beta_1(\beta_2-1)} \left(\frac{\beta_2-1}{\beta_2}\right)^{-\beta_2(\beta_2-1)} b^{-\beta_1(\beta_2-1)+\beta_2(\beta_2-1)} b'/\beta_2(\beta_1-\beta_2) \\
b_2^{\beta_1-\beta_2} &= \left(\frac{\beta_1-1}{\beta_1-\beta_2}\right)^{\beta_1-\beta_2} \left(\frac{\beta_1-1}{\beta_1-\beta_2}\right)^{\beta_1(\beta_2-1)} \left(\frac{\beta_1}{\beta_1-\beta_2}\right)^{-\beta_1(\beta_2-1)} \left(\frac{\beta_2-1}{\beta_2}\right)^{-\beta_2(\beta_2-1)} b^{(1-\beta_2)(\beta_1-\beta_2)} b'/\beta_2(\beta_1-\beta_2) \\
b_2^{\beta_1-\beta_2} &= \left(\frac{\beta_1-1}{\beta_1-\beta_2}\right)^{\beta_1-\beta_2} \left(\frac{\beta_1-1}{\beta_1-\beta_2}\right)^{\beta_1(\beta_2-1)} \left(\frac{\beta_1}{\beta_1-\beta_2}\right)^{-\beta_1(\beta_2-1)} \left(\frac{\beta_2-1}{\beta_2}\right)^{-\beta_2(\beta_2-1)} b^{(1-\beta_2)(\beta_1-\beta_2)} b'/\beta_2(\beta_1-\beta_2) \\
b_2 &= \frac{\beta_1-1}{\beta_1-\beta_2} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2(1-\beta_2)/(\beta_1-\beta_2)} \left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_2}}{b^{\beta_2-1}}
\end{aligned}$$

As  $\theta^{*\beta_1} = \frac{-b\beta_2}{\beta_1-\beta_2} b_1^{-1}$  then

$$\begin{aligned}
\theta^{*\beta_1} &= \frac{-b\beta_2}{\beta_1-\beta_2} \frac{\beta_1-\beta_2}{1-\beta_2} \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{\beta_2(\beta_1-1)/(\beta_1-\beta_2)} \frac{b^{\beta_1-1}}{b'^{\beta_1}} \\
\theta^{*\beta_1} &= \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{\beta_2(\beta_1-1)/(\beta_1-\beta_2)-1} \frac{b^{\beta_1}}{b'^{\beta_1}} \\
\theta^{*\beta_1} &= \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \frac{b^{\beta_1}}{b'^{\beta_1}} \\
\theta^* &= \left(\frac{\beta_1-1}{\beta_1}\right)^{-(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{(\beta_2-1)/(\beta_1-\beta_2)} \frac{b}{b'}
\end{aligned}$$

As  $(\beta_1 - \beta_2) b_2 \theta^{*\beta_2} = b' \theta^{*\beta_2} (\beta_1 - 1)$  then

$$\begin{aligned}
\theta^{*\beta_2-1} &= \frac{b'(\beta_1-1)}{\beta_1-\beta_2} b_2^{-1} \\
\theta^{*\beta_2-1} &= \frac{b'(\beta_1-1)}{\beta_1-\beta_2} \frac{\beta_1-\beta_2}{\beta_1-1} \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{-\beta_2(1-\beta_2)/(\beta_1-\beta_2)} \frac{b^{\beta_2-1}}{b'^{\beta_2}} \\
\theta^{*\beta_2-1} &= \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \left(\frac{1-\beta_2}{-\beta_2}\right)^{-\beta_2(1-\beta_2)/(\beta_1-\beta_2)} \frac{b^{\beta_2-1}}{b'^{\beta_2-1}} \\
\theta^{*\beta_2-1} &= \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2/(\beta_1-\beta_2)} \frac{b}{b'}
\end{aligned}$$

$$b_1 = \frac{1-\beta_2}{\beta_1-\beta_2} \left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{-\beta_2(\beta_1-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_1}}{b^{\beta_1-1}} \quad (30a)$$

$$b_2 = \frac{\beta_1-1}{\beta_1-\beta_2} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2(1-\beta_2)/(\beta_1-\beta_2)} \left(\frac{\beta_1-1}{\beta_1}\right)^{\beta_1(\beta_2-1)/(\beta_1-\beta_2)} \frac{b'^{\beta_2}}{b^{\beta_2-1}} \quad (30b)$$

$$\theta^* = \left(\frac{\beta_1-1}{\beta_1}\right)^{-(\beta_1-1)/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{(\beta_2-1)/(\beta_1-\beta_2)} \frac{b}{b'} \quad (30c)$$

$$\theta^{*\beta_2-1} = \left(\frac{\beta_1-1}{\beta_1}\right)^{-\beta_1/(\beta_1-\beta_2)} \left(\frac{\beta_2-1}{\beta_2}\right)^{\beta_2/(\beta_1-\beta_2)} \frac{b}{b'} \quad (30d)$$

Proof of Lemma 1

$F(p)$  may be written as  $F(p) = \max_{\{a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} p_{\tau_k} V(a_k)$  where current time is  $t = 0$  and  $\tau_k = \sum_{s=1}^k a_s$  is the sum of all harvest ages from the next harvest to the  $k^{th}$  harvest.

Since  $p$  is a GBM, multiplying  $p$  by  $\lambda$  implies that  $p_s$  is multiplied by  $\lambda$  for any  $s > t$ . Then  $F(\lambda p) = \max_{\{a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} (\lambda p_{\tau_k}) V(a_k)$

$$= \lambda \max_{\{a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} p_{\tau_k} V(a_k) = \lambda F(p)$$

Consequently, (9) can be written as

$$pF(1) = \max_s E e^{-rs} [p_s V(s) + p_s F(1)] \text{ or } pF(1) = \max_s [p e^{-\delta s} V(s) + p e^{-\mu s} F(1)]$$

$= p \max_s [e^{-\delta s} V(s) + e^{-\mu s} F(1)]$ . Thus the maximization which defines the optimal harvest age is independent of the price.

### Proof of Lemma 2

First, we will prove that  $F(p, p')$  is homogenous of degree one in  $(p, p')$ . Define  $\alpha_k$  as a dichotomous variable taking the value 1 if species  $P$  is planted after the  $(k-1)^{th}$  harvest or the value 0 otherwise. Similarly,  $\alpha'_k$  is a dichotomous variable taking the value 1 if species  $P$  is planted after the  $(k-1)^{th}$  harvest or 0 if species  $P$  is planted at that time. Thus,  $\alpha_k$  and  $\alpha'_k$  satisfy  $\alpha_k \in \{0, 1\}$ ,  $\alpha'_k \in \{0, 1\}$ , and  $\alpha_k \alpha'_k = 0$ .  $F(p, p')$  can be written as  $F(p, p') = \max_{\{\alpha_k, \alpha'_k, a_k\}} E \sum_{k=1}^{\infty} e^{-r\tau_k} (\alpha_k p_{\tau_k} V(a_k) + \alpha'_k p'_{\tau_k} V'(a_k))$ . Since  $p$  is a GBM, multiplying  $p_t$  by

$\lambda$  implies that  $p_{t'}$  is multiplied by  $\lambda$  for any  $t' > t$ . Then  $F(p, p')$  is homogenous of degree one in  $(p, p')$ . Considering equations (12a) and (12b), the functions  $G(p, p', a)$  and  $G'(p, p', a)$  are homogenous of degree one in  $(p, p')$  as well. Now, suppose that species  $P$  is currently planted; we want to prove that the optimal harvest age depends on  $\theta_s$  only, where  $\theta_s$  is the value of  $\theta$  when the stand is optimally cut. As  $G(p, p', a) = \max_{s \geq 0} E e^{-rs} [p_s V(s+a) + F(p_s, p'_s)]$ , then  $G(1, \theta, a) = \max_{s \geq 0} [e^{-\delta s} V(s+a) + E e^{-rs} \frac{p_s}{p} F(1, \theta_s)]$ . Note that  $\frac{p_s}{p}$  is the value at time  $s$  of a GBM with drift  $\mu$  and volatility  $\sigma$  whose value is 1 at time 0; thus  $\frac{p_s}{p}$  is independent of  $p$  (as  $\frac{p_s}{p}$  is a GBM with initial value 1). Since it is optimal to cut when

$\text{argmax}_s E e^{-rs} [p_s V(s+a) + F(p_s, p'_s)] = 0$ , the optimal harvest age depends on  $\theta$  only.

Let  $G(p, a) = pg(a)$  where  $g(a)$  is a function of the stand age to be determined. Except at harvest age,  $G(p, a)$  satisfies Bellman's equation  $E(dG(p, a)) = rG(p, a)da$  where  $EdG = G_a da + G_p p da + \frac{\sigma^2}{2} p^2 G_{pp} da + o(da)$ ,  $G_a = pg_a(a)$ ,  $G_p = g(a)$ , and  $G_{pp} = 0$ . Consequently  $EdG = pg_a(a) + \mu g(a) p da + o(da)$ , and Bellman's equation implies that  $g(a)$  must satisfy  $\delta g(a) = g_a(a)$  for  $a \in [0, a_f]$ , where  $\delta \equiv r - \mu$  and  $a_f$  is the optimal harvest age, the same for all harvests by Lemma 1. At any harvest,  $G(p, a)$  should satisfy the value-matching and smooth-pasting conditions linking  $G(p, a_f)$  and  $V(a_f)p + F(p)$ . Dividing both conditions by  $p$  yields  $g(a_f) = V(a_f) + f(\theta)$  and  $g_a(a_f) = V_a(a_f)$ .

Solving in the usual fashion, one finds that the forest and land value functions, together with  $c$ , a constant, and the optimal harvest age are determined by the equations in the Lemma.

### Proof Equation (18)

$G(p, p', a) = pg(\theta, a)$  must satisfy Bellman's equation  $EdG(p, p', a) = rG(p, p', a)da$ .

As  $dG = G_a da + G_p dp + \frac{1}{2} G_{pp} dp^2 + G_{p'} dp' + \frac{1}{2} G_{p'p'} dp'^2 + G_{p'p} dp dp' + o(da)$  where  $G_a(p, p', a) = pg_a(\theta, a)$ ,

$$G_p(p, p', a) = g(\theta, a) - \theta g_{\theta}(\theta, a),$$



$$\begin{aligned}
G_{pp}(p, p', a) &= \frac{\theta^2}{p} g_{\theta\theta}(\theta, a), \\
G_{p'}(p, p', a) &= g_{\theta}(\theta, a), \\
G_{p'p'}(a, p, p') &= \frac{1}{p} g_{\theta\theta}(\theta, a), \text{ and} \\
G_{pp'}(p, p', a) &= -\frac{\theta}{p} g_{\theta\theta}(\theta, a).
\end{aligned}$$

Then  $EdG(p, p', a)$

$$= pg_a da + \mu p g da + p \theta g_{\theta} (\mu' - \mu) da + \theta^2 g_{\theta\theta} \left( \frac{\sigma^2}{2} - \rho \sigma \sigma' + \frac{\sigma'^2}{2} \right) da + o(da)$$

and so  $\frac{\bar{\sigma}^2}{2} \theta^2 g_{\theta\theta} + \bar{\mu} \theta g_{\theta} - \delta g + g_a = 0$  where  $\bar{\sigma}^2 = \sigma^2 - 2\rho\sigma\sigma' + \sigma'^2$  and  $\bar{\mu} = \mu' - \mu$ . Note that  $\theta$  is a GBM with drift  $(\mu' - \mu) + \sigma^2 - \rho\sigma\sigma'$  and volatility  $\bar{\sigma}^2$ .

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