A Multivariate Stable Model for the Distribution of Portfolio Returns

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Abstract

In this paper we combine the appealing properties of the stable Paretian distribution to model the heavy tails and the GARCH model to capture the phenomenon of the volatility clustering. We assume the asset-returns to have a particular multivariate stable distribution, i.e., to be sub-Gaussian random vectors. In this way the characteristic function has a tractable expression and the density function can be recovered by using the Fast Fourier Transform and linear interpolation. A multivariate GARCH structure is then adopted to model the covariance matrix of the Gaussian vectors underlying the sub-Gaussian system. Finally, the model is applied to daily U.S. stock returns.

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1 Introduction

It is a very well known fact that time series returns on financial assets present features like volatility clustering, excess of kurtosis, i.e. the tails are “fatter” than implied by the normal distribution, and asymmetry.

To capture the phenomenon of heavy tails, Mandelbrot (1963) and Fama (1965) have been the first to introduce stable distributions to model the unconditional distribution of financial returns. After their pioneering works, stable distributions to model- unconditional and conditional- distribution of asset-returns have been matter of investigation in many studies. In academic literature stable distributions have been proposed as a model for many types of physical and economic system. There are several reasons for using a stable distribution to describe a system. The first is when for theoretical reasons we expect a non-Gaussian mode, e.g. hitting times for a Brownian motion yielding a Lévy distribution; see Feller (1971) for this and other examples. The second reason is the Generalized Central Limit Theorem which states that the only possible non-trivial limit of normalized sum of independent and identically distributed (i.i.d. hereafter) terms is stable. If we think the error terms in an econometric model as random variables representing the sum of external events not captured by the model, the use of the stable Paretian assumption seem reasonable. The third argument for modeling with stable distribution is empirical and related to the features of financial time series we presented before: heavy tails and skewness.

The classical objection moved against the stable assumption is that it has infinite variance. Empirical studies suggest the existence of the third or fourth moments for various financial data (cf. Pagan(1996)). However to reach to this conclusion, Hill (1975) or related tail estimators have been used, which are known to be not reliable for i.i.d. samples (cf. for example Mittnik and Rachev (1993); Mittnik et al. (1998a); Paolella (2001) ). Moreover, the same people who argue that the population is bounded and therefore must have a finite variance, routinely use the normal distribution or other ad hoc distributions with infinite support as a model for the same population. As pointed out in Nolan(2003) the only justification provided is that the normal distribution gives a usable description of the shape of the distribution. The variance is one measure of spread, as the scale parameter in the stable case is another. Nowadays for many practioners the variance is the measure of spread and any model with no finite variance is a priori rejected. If one would consider the variance just as the shape parameter of the Gaussian distribution, then the same can be done with the scale parameter of a stable distribution.

In this paper we give a time varying structure to the scale parameter of the distribution of the portfolio returns. To do this, we model with a multivariate GARCH the covariance matrix of the Gaussian vectors we assume to be underlying the stable system.

Our goal is to model the joint - conditional and unconditional - distribution of the vector of asset-returns of a portfolio assuming a multivariate stable distribution.

If we consider the return at time $t$ on a portfolio of, say, $k$ assets, a univariate conditional (e.g.
GARCH) or unconditional distribution can be fit to it. However, as the weight vector changes, the model has to be specified and fitted once again. If we work in a multivariate setting, the joint distribution if the returns can be directly used to compute the implied distribution of any portfolio.

One difficulty when working with stable distributions is that in general analytical expression for the density function is not available, and such a class of distributions is defined by its characteristic function (ch.f. hereafter). In the univariate setting one can use the inversion formula to recover the probability density function (p.d.f. hereafter). In this context the Fast Fourier Transform-based method (FFT hereafter) has been shown to perform particularly well when computing the density for a large number of data points (see Mittnik, Daganoglu and Chenyao (1999)). Unfortunately in the multivariate case the computation of the p.d.f. is even more complicated. A general expression for the characteristic function, in fact, involves computing an integral with respect to the so called spectral measure, i.e. a finite measure $\Gamma$ on the unit sphere $S_d \in \mathbb{R}^d$ with $d$ being the dimension of the multivariate stable distributed vector.

Modeling the joint distribution of the asset-returns under the stable assumption is a challenging task due to the complexity of the expression for the ch.f. in the general case and the consequent estimation problems. However, under some assumptions, it is possible to transform the problem of the multivariate pdf calculation to univariate pdf’s calculations. We briefly present an overview of the present literature on the topic.

Doganoglu and Mittnik (2006) use a stable multi-index model to generate a multivariate stable system. The basic idea underling the multi-index model is that there exists a set of common market factors such that each return series evolves as a linear combination of the factors plus an additive idiosyncratic noise process that is independent of these factors.

In this way the spectral measure is always discrete and given the independence between the factors and the disturbance component, the (multivariate) p.d.f. can easily be calculated using univariate p.d.f.’s. In Daganoglu, Hartz and Mittnik (2006) the same factor model is used and factors are considered to be conditionally varying. In both these papers it is clear how the assumption of a multivariate (symmetric or asymmetric) stable distribution for the asset returns reduces the systematic bias in Value-at-Risk computation compared with the normal assumption.

Nolan (2003) uses a multivariate stable elliptical distribution to model a multivariate series of financial returns. Given the particular expression of the ch.f. and exploiting the properties of the sub-Gaussian random variables, the parameters of the multivariate model are explicit functions of the parameters of the univariate series, which can be easily computed via ML estimation.

Lamantia, Ortobelli and Rachev (2005) present an extension of the EWMA RiskMetrics model considering elliptically distributed returns and examine several new methods based on different stable distributional hypotheses of return portfolio. Finally they discuss the applicability of temporal aggregation rules for each VaR and CVaR model proposed.

In this paper we use sub-Gaussian random vectors to generate a multivariate stable system.
Sub-Gaussians random vectors are a special case of stable random vectors. Some authors prefer the term “elliptically stable” as there are multiple meanings for sub-Gaussian in the probability literature, and they do not generally relate to stable distributions. We follow the notation in Smorodnitsky and Taqqu, 1994, and their definition of sub-Gaussian random vectors. This method allows us to have a tractable expression for the (multivariate) characteristic function and to express the scale parameter of the portfolio returns as a linear combination of the variances and covariances of the underlying Gaussian vectors. Under the sub-Gaussian hypothesis we are able to model the conditional and unconditional joint distribution of the asset returns. A multivariate GARCH model is introduced to describe the dynamics of the covariance matrix of the Gaussian vectors underlying the process. Given the computational complexity arising, we restrict our analysis to two dimensional case. The extension to a general $d$-dimensional case is theoretically straightforward.

The contribution of this paper comes from two sources. First we used a multivariate stable distribution to model the joint distribution of asset returns. Stable distributions have tails fatter than the Gaussian distribution, thus in our formulation we keep in account the phenomenon of excess of kurtosis. Second, under the sub-Gaussian hypothesis, we work with the covariance matrix of the underlying Gaussian vectors. Given that the scale parameter of the distribution of the portfolio is a linear combination of the entries of this matrix, we assume the variances and covariances to be time varying and introduce a multivariate GARCH structure. In this way we can capture the phenomenon of heteroskedasticity.

The second, indirect contribution of this paper is the extension, under the sub-Gaussian assumption, of the method in Mittnik, Doganoglu and Chenyao (1999) to the bivariate case. This allowed us to estimate the parameters of the model via Maximum Likelihood using the bivariate p.d.f.

The rest of the paper is organized as follows. In Section 2 we present the family of stable distributions in the univariate and multivariate case. In Section 3 sub-Gaussian random vectors are described. In Section 4 the sub-Gaussian hypothesis is used to model the joint distribution of the asset returns. In Section 5 we explain the way the bivariate stable p.d.f. was obtained. In Section 6 we provide an application to a bivariate stock return series. Section 7 concludes and proposes directions for future research.

2 Stable random variables and their properties

The theory of univariate stable distributions was essentially developed in the 1920’s and 1930’s by Paul Lévy and Alexander Yakovlevovich Khinchine. Recently, it was object of a monograph by Zolotarev (1986). This class of distribution nests two special distributions: the normal and the Cauchy. We now give two definitions of stable random variable. We follow the exposition in Samorodnitsky and Taqqu (1994).
Definition 1. (Samorodnitsky and Taqqu, 1994, p.2) A random variable $X$ is said to have a stable distribution if for any positive number $A$ and $B$, there is a positive $C$ and a real number $D$ such that

$$AX_1 + BX_2 \overset{d}{=} CX + D,$$

where $X_1$ and $X_2$ are independent copies of $X$.

A random variable $X$ is called symmetric stable if its distribution is symmetric, that is, $X$ and $-X$ have the same distribution.

Theorem 1. (Samorodnitsky and Taqqu, 1994, p.3)) For any stable random variable $X$, there is a number $\alpha \in (0, 2]$ such that the number $C$ in (1) satisfies

$$C^\alpha = A^\alpha + B^\alpha.$$

See Feller (1971) for a proof. The parameter $\alpha$ is called the index of stability or characteristic exponent. A stable random variable $X$ with index $\alpha$ is called $\alpha$-stable.

The second definition states that stable distributions are the only distributions that can be obtained as limit of normalized sums of i.i.d random variables. As mentioned before, is very intuitive to use a stable distribution when we assume the errors terms to be the sum of all external events that are not captured by the model.

Definition 2. (Samorodnitsky and Taqqu, 1994, p.5)) A random variable $X$ is said to have a stable distribution if it has a domain of attraction, i.e., if there is a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers $\{d_n\}$ and real numbers $\{a_n\}$, such that

$$\frac{Y_1 + Y_2 + \ldots Y_n}{d_n} + a_n \overset{d}{\to} X.$$

Stable random variables do not possesses a closed form for the p.d.f. an the distribution is defined via its ch.f. In the literature there are at least half a dozen different parameterizations. All involve different specifications of the ch.f. and are useful for various technical reasons. The parameterization most often used, e.g. Samorodnitsky and Taqqu, 1994, is the following:

the random variable $X$ is said to have a stable distribution if there are parameters $0 < \alpha \leq 2, c > 0, -1 < \beta < 1$ and $\mu$ real such that its ch.f. has the form

$$E[\exp\{i\theta X\}] = \begin{cases} 
\exp\{-c|\theta|^\alpha(1 - i\beta(\text{sign } \theta) \tan \frac{\pi \alpha}{2}) + i\mu \theta\} & \text{if } \alpha \neq 1, \\
\exp\{-c|\theta|(1 + i\beta \frac{2}{\pi}(\text{sign } \theta) \ln |\theta|) + i\mu \theta\} & \text{if } \alpha = 1,
\end{cases}$$

(4)

The parameters $\beta, c$ and $\mu$ are unique. Since (4) is characterized by these four parameters we will denote, as in Somorodnitsky and Taqqu, 1994, stable distributions by $S_\alpha(c, \beta, \mu)$ and write $X \sim S_\alpha(c, \beta, \mu)$. We also write $X \sim S_\alpha S$ when $X$ is symmetric $\alpha$-stable, i.e. when $\beta = \mu = 0$. It is easy to see that in (4) when $\alpha = 2$ and $\alpha = 1$, the ch.f. coincides with that of the normal
and Cauchy distribution respectively. Notice that for $\alpha = 2$, $\tan \frac{\pi \alpha}{2} = 0$, i.e. there is no “skewed normal” distribution within the stable family.

The complete name for stable distributions is “stable Paretian”. This expression reflects the fact that the asymptotic tail behavior is the same as that of the Pareto distribution, i.e. $S_\alpha$ it has power tails. The index $\alpha$ determines the thickness of the tails. When $\alpha = 2$, we have a Gaussian distribution. The smaller $\alpha$, the fatter the tails become.

For $0 < \alpha < 2$ the (fractional absolute) moments of $X \sim S_\alpha$ of order $\alpha$ do not exist. For $\alpha = 2$ all positive moment exist. This indeed coincides with the special cases Cauchy ($\alpha = 1$) and normal ($\alpha = 2$). The variance does not exist unless $\alpha = 2$.

One of the most important properties of the stable distribution is summability (or stability, hence part of the name). Summability means that the sum of independent stable random variables with the same index of stability $\alpha$, also follows a stable distribution. I.e. if

$$X_i \overset{\text{ind}}{\sim} S_\alpha(c, \beta_i, \mu_i), \text{ then } S = \sum_{i=1}^{n} X_i \sim S_\alpha(c, \beta, \mu)$$

where

$$\mu = \sum_{i=1}^{n} \mu_i, \quad c = (c_1^\alpha + \cdots + c_n^\alpha)^{1/\alpha} \text{ and } \beta = \frac{\beta_1 c_1^\alpha + \cdots + \beta_n c_n^\alpha}{c_1^\alpha + \cdots + c_n^\alpha}.$$  

The class of stable distribution is the only one which possesses the property of summability.

As in the univariate case, multivariate cumulative distribution functions or density functions are not usually known in closed form and therefore one works instead with the characteristic functions. We briefly introduce stable random vectors in a formal way.

**Theorem 2.** (Samorodnitsky and Taqqu, 1994, p. 57) A random vector $X = X_1, \ldots, X_d$ is said to be a stable random vector in $\mathbb{R}^d$ if for any positive number $A$ and $B$ there is a positive number $C$ and a vector $D \in \mathbb{R}^d$ such that

$$AX^{(1)} + BX^{(2)} =^d CX + D. \quad (5)$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies if $X$.

The vector $X$ is called **strictly stable** if (5) holds with $D = 0$ for any $A, B > 0$. The vector $X$ is called **symmetric stable** if it is stable and satisfies the additional relation

$$P\{X \in Q\} = P\{-X \in Q\}$$

for any Borel set $Q$ of $\mathbb{R}^d$.

The following theorems gives the distribution of any linear combination of the component of an $\alpha$-stable random vector.

**Theorem 3.** (Samorodnitsky and Taqqu, 1994, p. 58) Let $X = (X_1, \ldots, X_d)$ be a stable vector in $\mathbb{R}^d$. Then there is a constant $\alpha \in (0, 2]$ such that, in (5), $C = (A^\alpha + B^\alpha)^{1/\alpha}$. More over any linear combination of the components of $X$ of the type $Y = \sum_{k=1}^{d} b_k X_k$ is an $\alpha$-stable random variable.
As stated in Theorem 3, an $\alpha$-stable random vector possesses (as a Gaussian random vector) the appealing property that any linear combination of its component is $\alpha$-stable distributed. This can be a very useful property in portfolio theory because, under the assumption of a joint stable distribution of asset-returns, the returns of any portfolio of these assets is also $\alpha$-stable. The following theorem presents the expression of the characteristic function of the an $\alpha$-stable vector.

**Theorem 4.** (Samorodnitsky and Taqqu, 1994, 65) Let $0 < \alpha < 2$. Then $X = (X_1, \ldots, X_d)$ is an $\alpha$-stable random vector in $\mathbb{R}^d$ if and only if there exists a finite measure $\Gamma$ on the unit sphere $S_d$ of $\mathbb{R}^d$ and a vector $\mu^0$ in $\mathbb{R}^d$ such that:

(a) If $\alpha \neq 1$,

$$
\Phi_\alpha(\theta) = \exp \left\{ - \int_{S_d} |(\theta, s)^\alpha|(1 - i \operatorname{sign} (\theta, s) \tan \frac{\pi \alpha}{2}) \Gamma(ds) + i(\theta, \mu^0) \right\}. \quad (6)
$$

(b) If $\alpha = 1$,

$$
\Phi_\alpha(\theta) = \exp \left\{ \int_{S_d} |(\theta, s)| (1 + i \frac{2}{\pi} \operatorname{sign} (\theta, s) \ln |(\theta, s)|) \Gamma(ds) + i(\theta, \mu^0) \right\}. \quad (7)
$$

The pair $(\Gamma, \mu^0)$ is unique.

The vector $X$ in Theorem 4 is said to have spectral representation $(\Gamma, \mu^0)$. The measure $\Gamma$ is called the spectral measure of the $\alpha$-stable random vector $X$.

To define the characteristic function of a symmetric $\alpha$-stable random vector a necessary and sufficient condition is that $\mu^0 = 0$ and $\Gamma$ is a symmetric measure on $S_d$ (i.e. $\Gamma(Q) = \Gamma(-Q)$ for any Borel set $Q$ in $S_d$).

**Theorem 5.** (Samorodnitsky and Taqqu, 1994, 73) $X$ is a symmetric $\alpha$-stable vector in $\mathbb{R}^d$ with $0 < \alpha < 2$ if and only if there exists a unique symmetric finite measure $\Gamma$ on the unite sphere $S_d$ such that

$$
E[\exp\{i(\theta, X)\}] = \exp \left\{ - \int_{S_d} |(\theta, s)|^\alpha \Gamma(ds) \right\}. \quad (8)
$$

$\Gamma$ is the spectral measure of the symmetric $\alpha$-stable random vector $X$.

### 3 Sub-Gaussian Random Vectors

In this section we introduce the concept of Sub-Gaussian random vectors and exploit their properties to model the joint distribution of the asset-returns. They are a special case of symmetric $\alpha$-stable random vectors, but their spectral measure is always discrete and this allows us to have a tractable expression for the characteristic function. We start by presenting a useful characterization of the symmetric $\alpha$-stable random variables. The following result shows that one can always transform a $S\alpha'S$ random variable into a $S\alpha S$ random variable, for any $0 < \alpha < \alpha'$. 


Proposition 1. Let $G \sim S_{\alpha'}(c,0,0)$ with $0 < \alpha' \leq 2$ and let $0 < \alpha < \alpha'$. Let $A$ be an $\alpha/\alpha'$-stable random variable totally skewed to the right with Laplace transform

$$E[\exp\{-\gamma A\}] = \exp\{-2\gamma^{\alpha/\alpha'}\}, \quad \gamma > 0,$$

i.e. $A \sim S_{\alpha/\alpha'}\left(2(\cos \frac{\pi \alpha}{2\alpha'})^{\alpha'/\alpha}, 0, 0\right)$, and assume $G$ and $A$ to be independent.

Then

$$X = A^{1/\alpha'} G \sim S_{\alpha}(c \cdot 2^{1/\alpha'}, 0, 0).$$

If we consider now the particular case where $\alpha' = 2$, then $G$ becomes a zero mean Gaussian random variable; if the variance of $G$ is $\sigma^2$ then we have, taking the different scaling convention for normal and stale random variables into account, $G \sim S_2(\sigma/\sqrt{2}, 0, 0)$. By Proposition 1

$$X = A^{1/2} G \sim S_{\alpha}(\sigma, 0, 0)$$

This shows that every $S_{\alpha}$ random variable is conditionally Gaussian.

A result similar to Proposition 1 holds for skewed $G \sim S_{\alpha'}(c', \beta', 0)$ if $\alpha' \neq 1$, namely

$$X = A^{1/\alpha'} G \sim \begin{cases} S_{\alpha}(c, \beta, 0) & \text{if } a \neq 1 \\ S_1(c, 0, \mu) & \text{if } a = 1 \end{cases}$$

The random variable $X$ is skewed if $\alpha \leq 1$ and has non-zero shift vector if $\alpha = 1$. The parameters in its distribution are complicated functions of $c'$ and $\beta'$ (Hardin Jr. 1984); however, in the special case $\alpha' < 1$ and $\beta' = 1$, the parameter $\beta$ equals 1. As $\alpha'$ approaches 2, that is the Gaussian case, the parameter $\beta$ diminishes and when $\alpha = 2$, $\beta$ has no effect. Thus we will not be able to generate an asymmetric $\alpha$-stable random variable when $\alpha' = 2$.

The result of Proposition 1 can be extended to random vectors $X$ as follows. Choose

$$A \sim S_{\alpha/2}\left(2(\cos \frac{\pi \alpha}{4})^{2/\alpha}, 1, 0\right) \quad (9)$$

with $\alpha < 2$ so that its Laplace transform is

$$E[e^{-\gamma A}] = e^{-(2\gamma)^{\alpha/2}}, \quad \gamma > 0. \quad (10)$$

Let

$$G = (G_1, \ldots, G_d) \quad (11)$$

be a zero mean Gaussian vector in $\mathbb{R}^d$ independent of $A$. Then the random vector

$$X = (A^{1/2} G_1, \ldots, A^{1/2} G_d) \quad (12)$$

has a $S_{\alpha}$ distribution in $\mathbb{R}^d$ because, for any real numbers $b_1, \ldots, b_d$ the linear combination

$$\sum_{k=1}^d A^{1/2} G_k = A^{1/2} \sum_{k=1}^d G_k$$

is a $S_{\alpha}$ random variable (Proposition 1) and hence by Theorem 2.1.5 (report it or refer to Samorodnitsky and Taqqu 1994) $X$ is $S_{\alpha}$.

The random variable $X \sim N(0, \sigma^2)$ has ch.f. $\varphi_X(t) = \exp\{-\frac{1}{2} \sigma^2 t^2\}$. Using different notation we can write $X \sim S_2(c, 0, 0)$ and $\varphi_X(t) = -c^2 t^2 \sigma^2$ so that $c^2 = \frac{1}{2} \sigma^2$ and we can express the scale parameter $c = \frac{\sigma}{\sqrt{2}}$
Definition 3. (Samorodnitsky and Taqqu, 1994, p. 78) Any vector \( X \) distributed as in (12) is called a sub-Gaussian \( S\alpha S \) random vector in \( \mathbb{R}^d \) with underlying Gaussian vector \( G \). It is also said to be subordinated to \( G \).

The following proposition provides the special structure of the characteristic function of a sub-Gaussian random vector.

Proposition 2. The sub-Gaussian symmetric \( \alpha \)-stable random vector \( X \) defined in (12) has characteristic function

\[
E[\exp\{i \sum_{k=1}^{d} \theta_k X_k\}] = \exp\{-|\sum_{i=1}^{d} \sum_{j=1}^{d} \theta_i \theta_j \sigma_{ij}|^{\alpha/2}\},
\]

where \( \sigma_{ij} = E[G_i G_j] \) and \( \sigma_i^2 = E[G_i^2], i, j = 1, \ldots, d \) are the covariances and variances of the underlying Gaussian random vectors \( (G_1, \ldots, G_d) \).

When working in a multivariate setting, the ability to define (and thus to model) the dependency structure among assets is of fundamental importance. The covariance function is an extremely powerful tool in the study of Gaussian random elements, but it is not defined when \( \alpha < 2 \). The covariation is designed to replace the covariance when \( 1 < \alpha < 2 \). Unfortunately it lacks some of the desirable properties of the covariance.

Definition 4. (Samorodnitsky and Taqqu, 1994, p.87) Let \( X_1 \) and \( X_2 \) be jointly \( S\alpha S \) with \( \alpha > 1 \) and let \( \Gamma \) the spectral measure of the random vector \( (X_1, X_2) \). The covariation of \( X_1 \) on \( X_2 \) is the real number

\[
[X_1, X_2]_\alpha = \int_{S_2} s_1 |s_2|^{\alpha-1} \text{sign}(s_2) \Gamma(ds)
\]

With sub-Gaussians random vectors things are a bit simpler, and the following result enables us to have a close form for the entries of the covariation matrix.

Proposition 3. Consider positive random variable \( A \) as defined (9) independent of the vector \( G = (G_1, \ldots, G_n) \) as defined in (11) with \( \sigma_{ij} = E[G_i, G_j] \) and \( \sigma_i^2 = E[G_i^2], i, j = 1, \ldots, d \). The vector \( X = (X_1, \ldots, X_n) \), with \( X_i = A^{1/2} G_i, \ i = 1, \ldots, n \) is sub-Gaussian. We have that

\[
[X_i, X_j] = \sigma_{ij} \sigma_j^{(\alpha-2)}.
\]

Notice that \( [X_i, X_j]_\alpha = [X_j, X_i]_\alpha \) if \( \sigma_i^2 = \sigma_j^2 \). The covariation between two \( \alpha \)-stable random variable is, in fact, generally not symmetric in its arguments and Proposition 3 provides a sufficient condition for the symmetry in the arguments.

4 Sub-Gaussian random vectors and asset-returns

We introduced in the previous section the sub-Gaussian random vectors. The aim in this section is to use their properties to model the joint distribution of the vector of asset-returns of a portfolio.
As mentioned before, when working with multiple assets, it is important to consider the dependence structure among them. After Engle’s (1982) seminal paper, ARCH and GARCH models have been extended to the multivariate case in many different ways, so that the covariance structure among assets is time-varying. Stable random vectors however do not possess a covariance matrix. It would seem reasonable to replace the covariance matrix by the covariation matrix.

From Proposition 3 we have that

$$[X_i, X_j]_\alpha = \sigma_i^\alpha$$ and $$[X_i, X_j]_\alpha = \sigma_{ij}^\alpha \sigma_j^2.$$

One idea would be to introduce a multivariate GARCH structure in the covariation matrix. Its entries are non-linear functions of the covariances of the underlying Gaussian vectors generating the sub-Gaussians vector and do not have a direct interpretation. Moreover, the covariation matrix is not symmetric, leading to a even less clear comprehension of its meaning. Under the sub-Gaussian hypothesis the scale parameter of the distribution of the portfolio returns is a linear function of the covariance matrix of the underlying Gaussian vectors.

Let $A$ be a totally skewed $\alpha$-stable random variable as defined in (9), $0 < \alpha < 2$. Let $G_t = (G_{1t}, \ldots, G_{Nt})$ be a conditionally zero mean Gaussian vector independent of $A$ for every $t = 1, \ldots, T$, i.e.

$$G_t|\Pi_{t-1} \sim N_N(0, \Sigma_t), \quad \Sigma_{ij,t} = \sigma_{ij,t}, \quad i, j = 1, \ldots, N, \quad t = 1, \ldots, T$$

where $\sigma_{ij,t} = E[G_{it}G_{jt}]$.

Consider the vector of asset-returns at time $t$, $R_t, t = 1, \ldots, T$. Define $\epsilon_t = R_t - \mu_t$ the vector of demeaned returns.

**Assumption 1.** The vector

$$\epsilon_t = (X_{1t}, \ldots, X_{Nt}), \quad t = 1, \ldots, T$$

is a sub-Gaussian S\alpha S random vector with underlying Gaussian vector $G_t$; i.e.

$$\epsilon_t = (A^{1/2}G_{1t}, \ldots, A^{1/2}G_{Nt}), \quad t = 1, \ldots, T.$$

Under Assumption 1 we are able to define the distribution of the portfolio returns.

**Proposition 4.** Let $R_t = (R_{1t}, \ldots, R_{Nt})$ be a vector of asset returns at time $t, t = 1, \ldots, T$. Denote by $P_t$ the return of the portfolio at time $t$, i.e. $P_t = \sum_{i=1}^N \omega_i R_{it}$ with $\omega = (\omega_1, \ldots, \omega_N)$ representing the portfolio weights. If Assumption 1 holds, then

$$P_t|\Pi_{t-1} \sim S_{\alpha}(\sigma_t, 0, \omega'\mu),$$

with $\sigma_t^2 = \omega'\Sigma_t\omega$. 

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Note that $\alpha$ is not the characteristic exponent of the distribution of the portfolio returns, but the characteristic exponent of the multivariate stable distribution of the asset-returns. Thus estimating $\alpha$ directly from the distribution of $P_t$ would be a mistake. The stability index in fact determines the joint distributions of the asset-returns and does not depend on the way the portfolio is constructed. To estimate it we need to work with the characteristic function of the asset-returns, which has representation as in (13).

We are able to write the (unconditional and conditional) distribution of the portfolio returns as being $\alpha$-stable distributed in which the scale parameter is a direct function the the variance-covariance matrix $\Sigma$ (or $\Sigma_t$ in the conditional case) of the underlying Gaussian vectors generating the sub-Gaussian vectors. In this way we can easily model the dynamics for $\Sigma_t$ using a multivariate-GARCH model and in thus the dynamics of the scale parameter of the conditional distribution of the portfolio returns.

5 Efficient calculation of the $S\alpha S$ PDF’s

As has been said previously, there is no closed form expression for the density of a multivariate stable distribution. Only the characteristic function is known. When working with sub-Gaussian random vectors the ch.f. has a tractable expression. Starting from the ch.f. it is possible, via the inversion formula, to recover the multivariate pdf. To do this we extended to the bivariate setting the method of Mittnik et al. (1999).

In the univariate case, the $S\alpha S$ p.d.f. can be written as

$$p(x; \alpha, c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \varphi(t) dt. \tag{17}$$

which is the inversion formula to recover the p.d.f. when the ch.f. is known. In the bivariate case the formula becomes

$$p(x, y; \alpha, c) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ixt - iyt} \varphi(t_1, t_2) dt_1 dt_2. \tag{18}$$

In the univariate setting Mittnik et al. (1999) derives the p.d.f. directly as the Fourier transform of the ch.f. in (4). We extend their method to the bivariate case.

The FFT is an efficient way of computing the Fourier transform. The integral in (18) will be calculated in a lattice of $N_1 \times N_2$ equally-spaced points with distance $h_1$ and $h_2$, namely $x_k = (k - 1 - (N_1/2))h, k = 1, \ldots, N_1$ and $y_l = (l - 1 - (N_2/2))h, l = 1, \ldots, N_2$. Letting $t_1 = 2\pi u, t_2 = 2\pi v$, (18), becomes

$$p\left(\left(k - 1 - \frac{N_1}{2}, l - 1 - \frac{N_2}{2}\right)\right) = \int_{-\infty}^{\infty} \varphi(2\pi u, 2\pi v)e^{-i2\pi u(k - 1 - \frac{N_1}{2})h_1 - i2\pi v(l - 1 - \frac{N_2}{2})h_2} du dv. \tag{19}$$

The double integral in (19) can be approximated by using the rectangle rule for the lattice created
with the \( N_1 \times N_2 \) points with spacing \( s_1 \) and \( s_2 \), i.e.
\[
p \left( \left( k - 1 - \frac{N_1}{2}, l - 1 - \frac{N_2}{2} \right) \right) \approx s_1 s_2 \sum_{n_1=1}^{N_1} \sum_{n_2=2}^{N_2} \varphi \left( 2\pi s_1 (n_1 - 1 - \frac{N_1}{2}), 2\pi s_2 (n_2 - 1 - \frac{N_2}{2}) \right) \times \\
\exp \left\{ -i2\pi(n_1 - 1 - (N_1/2))(k - 1 - (N_1/2))h_1 s_1 - i2\pi(n_2 - 1 - (N_2/2))(l - 1 - (N_2/2))h_2 s_2 \right\}.
\]
(20)

By setting in (20) \( s_1 = (h_1 N_1)^{-1} \) and \( s_2 = (h_2 N_2)^{-1} \), we obtain the approximation
\[
p \left( \left( k - 1 - \frac{N_1}{2}, l - 1 - \frac{N_2}{2} \right) \right) \approx s_1(-1)^{k-1-(N_1/2)} s_2(-1)^{l-1-(N_2/2) \times \\
\sum_{n_1=1}^{N_1} \sum_{n_2=2}^{N_2} (-1)^{n_1+n_2-2} \varphi \left( 2\pi s_1 (n_1 - 1 - \frac{N_1}{2}), 2\pi s_2 (n_2 - 1 - \frac{N_2}{2}) \right) \times \\
\exp \left\{ (-i2\pi(n_1 - 1)(k - 1))/N_1 - i2\pi(n_2 - 1)(l - 1)/N_2 \right\}.
\]
(21)

The summation in (21) is computed by applying FFT to the sequence
\[
(-1)^{n_1+n_2-2} \varphi \left( 2\pi s_1 (n_1 - 1 - \frac{N_1}{2}), 2\pi s_2 (n_2 - 1 - \frac{N_2}{2}) \right).
\]
The \((k^{th}, l^{th})\) element of the resulting sequence is normalized by
\[
s_1(-1)^{k-1-(N_1/2)} s_2(-1)^{l-1-(N_2/2)}
\]
to obtain the pdf value for each lattice point.

Along with the method proposed by Mittnik et al. (1999a), the procedure to obtain pdf values for irregularly-spaced data consists of two steps. First we specify a lattice using two equally spaced grids covering the range of data and compute the pdf on the lattice of points. This can be done in two ways. One way is to use the Matlab function \texttt{fft2} which we exploit to compute the two dimensional Fourier transform on the \( N_1 \times N_2 \) matrix of the ch.f. calculated on the lattice. The other way is to vectorize the previous matrix containing the ch.f and use the Matlab function \texttt{fft}, which computes the one dimensional Fourier transform via FFT to recover the ch.f.. Neither of the two ways differs from the other in term of computational time, so the first approach was adopted.

In the second step we use two dimensional linear interpolation to the data points falling between the lattice values. To accomplish this we used the Matlab function \texttt{fft2}. Mittnik, Doganoglu and Chenyao (1999) suggest that for \( 1.6 < \alpha < 1.9 \), which are values typically of financial data, setting \( h = 0.01 \) and \( N = 2^{13} \) leads to a fast and sufficient accurate approximation. Unfortunately, using the same values for these tuning parameters in the bivariate case, increase in a notable way the computational burden. In this case greater speed is more desirable than greater accuracy. We set \( h_1 = h_2 = 0.04 \) and \( N_1 = N_2 = 2^{10} \). For this reason we can consider only a range of data belonging to the interval \([-10, 10]\). An ad hoc Matlab interpolation routine for this purpose would reduce the number of necessary calculation and allow us to increase the accuracy. Building such a routine is a priority for the near future.
6 Application to stock market

We model daily return data from the General Motors and Johnson & Johnson stock using a sample form 01/01/1990 to 29/12/1995 implying 1517 observations downloaded from Yahoo Finance. Continuously compounded percentage returns are considered, i.e. daily returns are measured by log-differences of closing pricing multiplied by 100. To obtain the series of the demeaned returns we subtracted to each series its mean. Given the insignificant autocorrelation we assumed the mean to be constant over time for both series. Sample path and marginal kernel density estimates are given in Figure B. Descriptive statistics for the univariate series are given in Table 1 and the result for the univariate $\alpha$-stable model estimation are presented in Table 2. From a preliminary exploratory analysis we see that both series are leptokurtik and present a (weak) asymmetry. When estimating the parameters of the stable distribution the thickness of the tails is captured by the the estimated value of $\alpha$, which, for both series, is statistically different from 2, the stability index of a Gaussian distribution. In both series the estimate for $\beta$ is not significant hence we conclude the there is no asymmetric distribution in our case.

This preliminary analysis suggests that a Gaussian model for the joint distribution of the returns may not seem appropriate. The stability index of the two series is different from 2 and the asymmetry is not significant. Our next step is hence to fit a multivariate sub-Gaussian model to the unconditional joint distributions of the series. Table 3 presents the estimates for the parameters of the model.

The index $\alpha$ is 1.86 (0.02) rejects the normal hypothesis $\alpha = 2$ and indicates that the tails of the distributions are heavier than implied by the normal distribution. An evaluation of the log-likelihood functions (reported for all the models in Table 4)) with the parameters estimates favors the stable model. The value of the standard likelihood-ratio test statistic for normal versus stable model, given by

$$LR_{N,S} = 2 - (Loglik_{normal} - Loglik_{stable}) = 73.4$$

exceeds the 99%-critical value of the $\chi^2_1$ distribution, which is 6.635. This means a clear rejection of the null hypothesis “$\alpha = 2$ ”.

For the conditional distribution we decided to use, amongst the numerous multivariate GARCH proposed in literature (see for example Bauwens et al. (2006) for an overview), the BEKK model of Engle and Kroner (1995). This model is a particular case of the VEC model proposed by Bollerslev et al. (1988) with a set of restrictions on the parameters to ensure the positivity of the covariance matrix.

The VEC(1,1) model is defined as

$$h_t = C + A\eta_{t-1} + G h_{t-1}$$

(22)

where

$$h_t = vech(\Sigma_t)$$
and

$$\eta_t = \text{vech}(\epsilon_t')$$

and \text{vech} denotes the operator that stacks the lower triangular portion of a \(n \times N\) matrix as a \(N(N+1)/2 \times 1\) vector. \(A\) and \(G\) are square parameter matrices of order \(N(N+1)/2\) and \(C\) is a \(N(N+1)/2 \times 1\) parameter vector. \(\Sigma_t\) is the covariance matrix of the demeaned returns \(\epsilon_t\), \(t = 1, \ldots, T\). If the matrices \(A\) and \(G\) are assumed to be diagonal, the model becomes a diagonal VEC.

The BEKK(1,1,K) is defined as

$$\Sigma_t = C^* C^* + \sum_{k=1}^K A_k^* \epsilon_t' \epsilon_{t-1} + \sum_{k=1}^K G_k^* \Sigma_{t-1} G_k^*$$

(23)

where \(C^*, A_k^*\) and \(G_k^*\) are \(N \times N\) matrices but \(C\) is upper triangular; \(N\) is the number of assets. In many practical applications, \(k = 1\) is the standard choice. In this specification the model is identified if the diagonal elements of \(C\), as well as the top left element of the matrices \(A\) and \(G\) are restricted to be positive. We will impose these restrictions in the application below. The number of parameters in a BEKK(1,1,1) model is \(N(5N+1)/2\), in our case 11. To reduce this number we impose a diagonal BEKK model, i.e. \(A^*\) and \(G^*\) in (23) are diagonal matrices. The number of parameters to be estimated is now 7. The VEC model is covariance stationary\(^2\) if \(\rho(D) < 1\), where \(\rho(D)\) is the largest eigenvalue of the matrix \(D = A + G\). If we consider the diagonal BEKK model as a diagonal VEC model with additional restrictions on the parameters, we can rewrite

\[
\begin{pmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
0 & 0 & A_{33}
\end{pmatrix}
= \begin{pmatrix}
A_{11}^{*2} & 0 & 0 \\
0 & A_{22}^{*2} & 0 \\
0 & 0 & A_{33}^{*2}
\end{pmatrix},
\begin{pmatrix}
G_{11} & 0 & 0 \\
0 & G_{22} & 0 \\
0 & 0 & G_{33}
\end{pmatrix}
= \begin{pmatrix}
G_{11}^* & 0 & 0 \\
0 & G_{22}^* & 0 \\
0 & 0 & G_{33}^*
\end{pmatrix}
\]

Then the stationarity condition becomes

$$\max(A_{11}^{*2} + G_{11}^*; A_{11}^* A_{22}^* + G_{11}^* G_{22}^*; A_{22}^{*2} + G_{22}^{*2}) < 1.$$  \hspace{1cm} (24)

We estimated a BEKK(1,1,1) model for the bivariate series of the residuals under the Normal and Stable assumptions. Estimation results are are presented in Table 3. The parameters have been estimated via Maximum Likelihood. For the stable BEKK the p.d.f. has been recovered with the method described in Section 5. To estimate the normal BEKK model, and to plot the contour plot of the kernel density, we used the Matlab toolbox \textit{UCSD garch} written by Kevin Sheppard and available under the BSD style license\(^3\).

Likelihood-based goodness of fit criteria, as well as the value of the log-likelihood at its maximum are shown in Table 4.

---

\(^2\) For this condition see Engle and Kroner (1995)

\(^3\) http://www.kevinsheppard.com/research/ucsd_garch/ucsd_garch.aspx
Not surprisingly stable model performs better in term of likelihood maximization, whether for the unconditional and the unconditional distributions. Normal distribution is the worst performer also with respect to the Akaike (1973) and Schwarz (1978) information criteria we adopted. This is due to the excess of kurtosis which characterizes the two series and the difference would be even stronger with more leptokurtic series.

The maximum likelihood estimates for the stable, stable-GARCH(1,1) and Normal-GARCH(1,1), are reported in Table 3. The estimates for the Normal-GARCH and the stable-GARCH are quite similar. Both models are covariance stationary, being the largest eigenvalue of \( A + G \) in the VEC representation smaller than 1. The estimated value of the stability index in the two bivariate stable models is slightly different. This is probably due to the not perfect convergence of the iteration process of the algorithm we have encountered while estimating the parameters.

7 Conclusion

In this paper we proposed a multivariate stable model for the distribution of the asset returns of a portfolio. The class of stable distributions possesses a parameter, the stability index \( \alpha \), which determines the thickness of the tails and thus allows us to keep in account the phenomenon of excess of kurtosis. Under the hypothesis that the asset returns have a sub-Gaussian joint distribution, the joint characteristic function possesses a tractable expressions and this allowed us to estimate the parameters via the likelihood function maximization. The joint density function was recovered by extending to the bivariate case the method of Mittnik, Doganoglu and Chenyao (1999). Moreover, given the particular expression of the characteristic function, we can introduce a multivariate GARCH model for the covariance matrix of the Gaussian vectors underlying the sub-Gaussian system. The scale parameter of the \( \alpha \)-stable distributed portfolio returns is in fact a linear combination of this covariance matrix. In this way we keep in account an other feature typical of financial time series: the heteroskedasticity.

An application to two daily returns for the General Motors and the Johnson & Johnson stock has been presented. The stable model performs better in terms of goodness of fit when compared with the normal, for the conditional and unconditional distribution.

The use of multivariate stable distribution is still a challenge. Our formulation gave us a rather simple expression for the ch.f., but still the computational burden is notable. Building specific routines to invert the ch.f. to obtain the multivariate stable p.d.f. is a priority for our future research, as well as the extension to higher dimensions.

In the BEKK, even in its diagonal formulation, the number of parameters is high and this model is rarely used when the number of series is larger than 3 or 4. Future research will focus on other more parsimonious parameterization.

The calculation of the Value-at-Risk of a portfolio would be a good application to check the appropriateness of the stable assumption for risk management.
Together with volatility clustering and excess of kurtosis, asymmetry is an other important feature present in financial series. Our model is unfortunately restricted to be asymmetric. Developing a multivariate stable model which has a tractable expression and allows for asymmetry is without any doubts a challenging task and it is left for future research.

References


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[23] Lamantia F., Ortobelli S., Rachev S. Risk Management an Dynamic Portfolio Selection with Stable Paretian Distributions. Working Paper


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A Appendix

Proof of Proposition 1
The proof is analogous to the one of Proposition 1.3.1 in Samorodnitsky and Taqqu, 1994. For a real $\theta$ we have

$$E[\exp\{-i\theta X\}] = E[\exp\{i\theta A^{1/\alpha'} X\}]$$
$$= E[E[\exp\{i\theta A^{1/\alpha'} X\}|A]]$$
$$= E[\exp\{-|\theta|^{\alpha} e^{\alpha'} A\}]$$
$$= \exp\{-|(2\theta|^{\alpha} e^{\alpha'})^{\alpha/\alpha'}\}$$
$$= \exp\{-|\theta|^{\alpha}(c \cdot 2^{1/\alpha'})^{\alpha}\}.$$  

We recognize the ch. f. of an $\alpha$-stable random variable. Then

$$X \sim S_\alpha(c \cdot 2^{1/\alpha'}, 0, 0).$$  

Proof of Proposition 2
The proof is analogous to the one of Proposition 2.5.2 in Samorodnitsky and Taqqu, 1994. By conditioning on $A$ we have that

$$E[\exp\{i \sum_{i=1}^{d} \theta_i X_k\}] = E[E[\exp\{i A^{1/2} \sum_{i=1}^{d} \theta_i X_k\}|A]]$$
$$= E[\exp\{-A(\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} \theta_i \theta_j)\}]$$
$$= \exp\{-\left(\sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} \theta_i \theta_j\right)^{\alpha/2}\}.$$  

which follows because $E[\exp\{-\gamma A\}] = \exp\{-2^{\alpha/2}\}, \gamma > 0.$

Proof of Proposition 3
The proof is analogous to the one of Example 2.7.4 in Samorodnitsky and Taqqu, 1994. Since $X$ has characteristic function as given in 2, we see that the scale parameter $\sigma(\theta_i, \theta_j)$ of $Y = \theta_i X_i + \theta_j X_j, i, j = 1, \ldots, n,$ satisfies

$$\sigma^\alpha(\theta_i, \theta_j) = (\theta_i^2 \sigma_i^2 + 2\theta_i \theta_j \sigma_i \sigma_j + \theta_j^2 \sigma_j^2)^{\alpha/2}.$$  

Equivalent to Definition 4 is the following

Definition 5. (Samorodnitsky and Taqqu, 1994, p.88) The covariation $[X_1, X_2]_\alpha$ of $(X_1, X_2)$ is

$$[X_1, X_2]_\alpha = \frac{1}{\alpha} \frac{\partial \sigma^\alpha(\theta_1, \theta_2)}{\partial \theta_1} \bigg|_{\theta_1=0, \theta_2=1}. \quad (25)$$  

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Using (25), we see that

\[ [X_i, X_j]_\alpha = \frac{1}{\alpha} \left. \frac{\partial \sigma^\alpha(\theta_i \theta_j)}{\partial \theta_i} \right|_{\theta_i=0, \theta_j=1} = \sigma_{ij} \sigma_j^{\alpha-2}. \]

**Proof of Proposition 4**

Consider the random variable \( Y_t \),

\[ Y_t = \sum_{i=1}^{N} \omega_i X_{it}. \]

If Assumption 1 holds we can write

\[ Y_t = A^{1/2} \sum_{i=1}^{N} \omega_i G_{it}. \]

Using the well know property of the Gaussian random variables

\[ \sum_{i=1}^{N} \omega_i G_{it} \mid \mathbb{I}_{t-1} \sim N(0, \omega^\prime \Sigma_t \omega) \]

or, with different notation

\[ \sum_{i=1}^{N} \omega_i G_{it} \mid \mathbb{I}_{t-1} \sim S_2(\frac{\sigma_t}{\sqrt{2}}, 0, 0), \quad \sigma_t^2 = \omega^\prime \Sigma_t \omega. \]

Then, by Proposition 1

\[ Y_t \mid \mathbb{I}_{t-1} \sim S_\alpha(\sigma_t, 0, 0) \]

and given that \( P_t = Y_t + \omega^\prime \mu \), using the properties of the \( \alpha \)-stable distribution

\[ P_t \mid \mathbb{I}_{t-1} \sim S_\alpha(\sigma_t, 0, \omega^\prime \mu). \]
B  Figures and Tables

Figure 1: Daily returns from January 1990 to December 1995

Figure 2: Kernel density estimates of General Motors (left) and Johnson&Johnson (right).
Figure 3: Contour plot for the General Motors - Johnson&Johnson

Table 1: Descriptive statistics of the General Motors - Johnson&Johnson returns.

<table>
<thead>
<tr>
<th>Covariance matrix</th>
<th>Mean</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen. Mot.</td>
<td>3.5782</td>
<td>0.4964</td>
<td>0.0240</td>
</tr>
<tr>
<td>J &amp; J</td>
<td>0.4964</td>
<td>2.2468</td>
<td>0.0763</td>
</tr>
</tbody>
</table>

Table 2: Result for the univariate asymmetric $\alpha$-stable model. Standard errors are expressed in parenthesis and were computed using the numeric approximation of the hessian matrix.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$c$</th>
<th>$\beta$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Motors</td>
<td>1.8746</td>
<td>1.2501</td>
<td>0.2884</td>
<td>0.0454</td>
</tr>
<tr>
<td></td>
<td>(0.0383)$^a$</td>
<td>(0.0306)$^a$</td>
<td>(0.1883)</td>
<td>(0.0517)</td>
</tr>
<tr>
<td>Johnson &amp; Johnson</td>
<td>1.8979</td>
<td>0.9946</td>
<td>0.0068</td>
<td>0.0789</td>
</tr>
<tr>
<td></td>
<td>(0.0325)$^a$</td>
<td>(0.0226)$^a$</td>
<td>(0.2292)</td>
<td>(0.0401)$^b$</td>
</tr>
</tbody>
</table>

$^a$significant at 1%; $^b$significant at 10%.
Table 3: General Motors - Johnson & Johnson multivariate GARCH(1,1) results. Standard errors are reported in parenthesis. For the stable models they were computed using the numeric approximation of the hessian matrix. Robust standard errors were computed for the normal-GARCH.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Stable</th>
<th>Normal-GARCH(1,1)</th>
<th>Stable-GARCH(1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.8635</td>
<td>-</td>
<td>1.8974</td>
</tr>
<tr>
<td></td>
<td>(0.0279)</td>
<td></td>
<td>(0.0252)</td>
</tr>
<tr>
<td>$\sigma_1^2$</td>
<td>1.5703</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0677)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_2^2$</td>
<td>0.9742</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0415)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{1,2}$</td>
<td>0.2170</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.0351)</td>
<td></td>
<td></td>
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<tr>
<td>$C_{1,1}$</td>
<td>-</td>
<td>0.2056</td>
<td>0.1592</td>
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<td></td>
<td></td>
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<td>(0.0421)</td>
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<td>0.0421</td>
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<td></td>
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<td>(0.0202)</td>
</tr>
<tr>
<td>$C_{2,2}$</td>
<td>-</td>
<td>0.2122</td>
<td>0.1938</td>
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<tr>
<td></td>
<td></td>
<td>(0.0629)</td>
<td>(0.0342)</td>
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<tr>
<td>$A_{1,1}$</td>
<td>-</td>
<td>0.1644</td>
<td>0.1056</td>
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<tr>
<td></td>
<td></td>
<td>(0.0469)</td>
<td>(0.0150)</td>
</tr>
<tr>
<td>$A_{2,2}$</td>
<td>-</td>
<td>0.1975</td>
<td>0.1578</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0360)</td>
<td>(0.0192)</td>
</tr>
<tr>
<td>$B_{1,1}^*$</td>
<td>-</td>
<td>0.9807</td>
<td>0.9800</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0159)</td>
<td>(0.0064)</td>
</tr>
<tr>
<td>$B_{1,1}$</td>
<td>-</td>
<td>0.9693</td>
<td>0.9529</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0128)</td>
<td>(0.0117)</td>
</tr>
</tbody>
</table>

$\rho(D) = 0.98$  $\rho(D) = 0.97$
**Table 4:** Likelihood-based goodness of fit. $K$ is the number of parameters in the model, $L$ is the value of the log-likelihood, $AIC = -2L + K$ and $BIC = -2L + G\log T$ are the Akaike and Schwarz information criteria respectively. $T$ is the number of observations.

<table>
<thead>
<tr>
<th>Model</th>
<th>$K$</th>
<th>$L$</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>3</td>
<td>-5863.4</td>
<td>11733</td>
<td>11749</td>
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<tr>
<td>Normal-GARCH(1,1)</td>
<td>7</td>
<td>-5781.5</td>
<td>11577</td>
<td>11614</td>
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<tr>
<td>Stable</td>
<td>4</td>
<td>-5826.7</td>
<td>11661</td>
<td>11683</td>
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<td>Stable-GARCH(1,1)</td>
<td>8</td>
<td><strong>5767.6</strong></td>
<td><strong>11511</strong></td>
<td><strong>11594</strong></td>
</tr>
</tbody>
</table>