PRICING AND HEDGING MULTI-ASSET OPTIONS
(INCOMPLETE PRELIMINARY VERSION, DO NOT QUOTE)

MIRET PADOVANI AND PAOLO VANINI

Abstract. The aim of this paper is to investigate the implications of stochastic correlations when pricing and hedging structured products written on multiple assets. A wide range of multi-asset options have emerged in the last years: rainbows, Himalayas, quantos, and spreads, just to name a few. The trickiness in pricing and hedging these options mainly stems from the volatilities of and pairwise correlations among their underlyings. For our analysis and for the sake of practicality, we take the specific example of a three-color best-of call rainbow option. We look at different pricing models both with constant and stochastic variance-covariance structures; for the latter, we give special attention to the multi-dimensional Ornstein-Uhlenbeck process, which has only recently been introduced in the derivative pricing literature. We then proceed to calibrate each model to data, perform sensitivity analyses, and determine appropriate hedging strategies. The question we ultimately want to answer is whether the introduction of stochastic correlations can lead to a model which is both theoretically meaningful and practically usable.

1. Introduction

The aim of this paper is to investigate the implications of stochastic correlations when pricing and hedging structured products written on multiple assets. We look at different pricing models both with constant and stochastic correlations. We then calibrate each model to data, perform sensitivity analyses, and determine appropriate hedging strategies. To our knowledge, such an extensive analysis on multi-asset products has not been carried out so far.

Correlations (as well as vol-of-vols) constitute primary market risks for a wide variety of multiple-asset options which have been structured in the last years. Just to name a few such products: Quantos, in which cashflows are calculated from an underlying in one currency and then converted into payment in another currency; Himalaya options, in which the best-performing underlying is taken out of the basket at specified sampling dates, leaving at the end just one underlying on which

Date: 30 September 2007.

Key words and phrases. Stochastic volatility; stochastic correlations; rainbow options; option sensitivities; model calibration; hedging strategies.

Miret Padovani, Swiss Banking Institute, University of Zurich;

E-mail address: padovani@isb.uzh.ch.

Paolo Vanini, Zürcher Kantonalbank and Swiss Banking Institute, University of Zurich;

E-mail address: vanini@isb.uzh.ch.

The authors are thankful for helpful comments and discussions to Aydin Akguen, Claudio Albanese, Umberto Cherubini, Marc Chesney, Silvan Ebnoether, Peter Schwendner and all participants of the Summer School on Frontiers in Financial Markets Mathematics in Bologna, Italy.

the payoff is determined;[1] Spread options, in which the payoff depends on the difference between the underlyings. One large class of multi-asset options is that of rainbow options, which essentially includes puts and calls written on the best- or worst-performing among n underlyings.[2] The pricing of all these products critically relies on the choice of correlation structure. For the sake of practicality, our study takes the specific example of a three-color best-of call rainbow option and focusses on equity derivatives.

Multi-asset options typically represent a cheaper alternative to hedge a risky position consisting of several assets, since trading only one rather than several options involves lower transaction costs. Nonetheless and with those advantages in mind, hedging multi-asset options poses great challenges precisely because of the correlation structure. Correlation risk is tricky to manage: correlations are not directly observable and relying on historical data could be misleading because of large market fluctuations. This is to be contrasted with forex derivatives, for which exchange rates are directly observed. Wystup (2002) illustrates how the correlation risk associated with forex basket options can be reduced to volatility risk using the interdependence of exchange rates; this, however, does not apply to other asset classes.

Usual business practice when dealing with rainbow options is to take constant correlation coefficients - and this despite empirical analyses unambiguously showing that "there is no such thing as constant volatility and correlation" (Taleb 1998). As proof of non constant correlations, we took monthly price data from Bloomberg on three S&P500 stocks - GS, GE, MER - from July 2004 to December 2005 and checked the correlation matrices for three subperiods of six months: despite the shortcoming of historical data just mentioned, Table 1 clearly illustrates that correlations among pairs of stocks can differ substantially from one semester to the other.

A rainbow option is intuitively more valuable the lower the correlations between its underlyings, because of a greater difference in the underlyings' movements. As correlations become negative and tend to -1, the option is guaranteed to be in the money in at least one of its assets and option value increases. There are a few early studies on the pricing of rainbow options given constant correlation coefficients. Stulz (1972) illustrates the pricing of calls and puts written on the maximum or minimum of two assets; whereas Johnson (1987) extends the results of Stulz to the case of $n > 2$ assets. In a relatively more recent paper, Rubinstein (1995) collects pricing formulas for a variety of rainbow options with constant correlations.

A few recent studies have taken the derivative pricing literature a step further and beyond constant volatilities and correlations. One line of research defines the covariance matrix as following a multi-dimensional Ornstein-Uhlenbeck process (Barndorff-Nielsen and Stelzer 2006; Pigorsch and Stelzer 2006). In a similar fashion, another equally recent line of research takes the multidimensional variance-covariance process as a superposition of Ornstein-Uhlenbeck processes; i.e. the variance is defined through a number of factors each having Ornstein-Uhlenbeck

[1]Overhaus (2002) discusses the importance of volatility and correlation in the pricing and sensitivity analysis of Himalaya options.

[2]Some authors employ the term 'rainbow option' to define virtually *any* option written on multiple underlyings. See, e.g., Wilmott (2006).

	GS	GE	MER
GS	1.0000	0.8164	0.9317
GE	0.8164	1.0000	0.6506
MER	0.9317	0.6506	1.0000

July 2004 - Dec 2004

	GS	GE	MER
GS	1.0000	-0.8626	0.5791
GE	-0.8626	1.0000	-0.3048
MER	0.5791	-0.3048	1.0000

Jan 2005 - June 2005

	GS	GE	MER
GS	1.0000	0.1087	0.9754
GE	0.1087	1.0000	0.2814
MER	0.9754	0.2814	1.0000

July 2005 - Dec 2005

Table 1. Correlation matrices for GS, GE, MER from July 2004 to December 2005

(one-dimensional) dynamics (Hubalek and Nicolato 2007). This paper is a contribution to this line of research.

The remainder of the paper is organized as follows. In Section 2, we address the pricing of a three-color best-of call option. We initially adapt pricing formulas for best-of rainbow options derived by Johnson (1987); we then proceed to price the option in the multi-dimensional Ornstein-Uhlenbeck case. In Section 3, we analytically compute and interpret option sensitivities, especially the deltas, gammas, cross-gammas, vegas, and correlation vegas. We then proceed to calibrate the models to data and determine hedging strategies for each model in Sections 4 and 5, respectively. We draw our conclusions and outline future research ideas in Section 6. Our aim is to check which model performs better and to quantify the tradeoff between applicability of a model and its theoretical meaningfulness. It is common wisdom in finance that models need to be sufficiently simple to be implementable, yet they also need to mirror observed data as closely as possible. Introducing increasing degrees of randomness in a model increases the number of parameters to be estimated and may make the model somehow prohibitive. Ultimately, the question we want to answer is: Can the introduction of stochastic correlations lead to a model which is both theoretically meaningful and practically usable?

2. Pricing "best-of" rainbow options

We take the specific example of a call option written on the best-performing of three underlying assets. The option's payoff is given by

$$\max[0, \max(S_1, S_2, S_3) - K],$$

where $S_i, i = \{1, 2, 3\}$, denotes the price of the ith underlying asset and K denotes the strike price.

The following subsections consider the cases of constant correlations and of stochastic correlations stemming from Ornstein-Uhlenbeck-based variance-covariance dynamics.

2.1. Pricing "best-of" rainbow options with constant correlations.

The risk-neutral dynamics of the three-component vector, $S_t \in \mathbb{R}^3$, containing the prices of the underlyings are given by

$$\frac{\mathrm{d}S_t}{S_t} = r\mathbb{I}_3 \mathrm{d}t + \sqrt{\Sigma_t}\mathrm{d}W_t^S, \tag{2.1}$$

where $r \in \mathbb{R}^+$ is the risk-free interest rate, $\mathbb{I}_3 \in \mathbb{R}^3$ is a vector of ones, and Σ_t is the 3×3 variance-covariance matrix. In this subsection, the variance-covariance matrix does not change over time, so that

$$\Sigma_t = \Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix},$$

where σ_i^2 is the variance of the ith stock price, ρ_{ij} is the correlation between assets i and j, and $\rho_{ij}\sigma_i\sigma_j$ is the covariance between assets i and j. In the next subsections, the variance-covariance matrix will be set to change over time according to some dynamic processes.

A valuation formula for a call on the maximum of n risky assets with strike price K and maturity date T has been derived by Johnson (1987). We adapt this valuation formula for the specific case with three risky assets. The pricing is done at $t = 0$ and no dividends are paid out.

The price C of the three-color best-of call is therefore

$$\begin{aligned} C \;=\; & S_1 \mathcal{N}_3\left(d_1(S_1, K, \sigma_1^2), d_1^\flat(S_1, S_2, \sigma_{12}^2), d_1^\flat(S_1, S_3, \sigma_{13}^2); \rho_{112}, \rho_{113}, \rho_{123}\right) \\ & + S_2 \mathcal{N}_3\left(d_1(S_2, K, \sigma_2^2), d_1^\flat(S_2, S_1, \sigma_{12}^2), d_1^\flat(S_2, S_3, \sigma_{23}^2); \rho_{212}, \rho_{213}, \rho_{223}\right) \\ & + S_3 \mathcal{N}_3\left(d_1(S_3, K, \sigma_3^2), d_1^\flat(S_3, S_1, \sigma_{13}^2), d_1^\flat(S_3, S_2, \sigma_{23}^2); \rho_{312}, \rho_{313}, \rho_{323}\right) \\ & - K\mathrm{e}^{-rT}\left[1 - \mathcal{N}_3\left(-d_2(S_1, K, \sigma_1^2), -d_2(S_2, K, \sigma_2^2), -d_2(S_3, K, \sigma_3^2), \rho_{12}, \rho_{13}, \rho_{23}\right)\right], \end{aligned}$$

where the superscript $\flat$ indicates that the interest rate is set to zero, $\mathcal{N}_3$ denotes the multivariate normal cumulative distribution function (cdf) with $n = 3$, [3]

$$d_1(S_i, K, \sigma_i^2) = \frac{\log \frac{S_i}{K} + \left(r + \frac{\sigma_i^2}{2}\right) T}{\sigma_i \sqrt{T}},$$

$$d_1^\flat(S_i, S_j, \sigma_{ij}^2) = \frac{\log(\frac{S_i}{S_j}) + \frac{\sigma_{ij}^2}{2} T}{\sigma_{ij} \sqrt{T}},$$

$$d_2(S_i, K, \sigma_i^2) = \frac{\log \frac{S_i}{K} + \left(r - \frac{\sigma_i^2}{2}\right) T}{\sigma_i \sqrt{T}},$$

$$\rho_{iij} = \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}},$$

$$\rho_{ijk} = \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}},$$

and

$$\sigma_{ij}^2 = \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2. \tag{2.2}$$

The summands in the formula above have an analogue interpretation to that in a Black-Scholes framework: each of the first three summands represents the discounted expected benefit of owning the option when either one of S_1, S_2, S_3 is the highest-performing stock; the fourth summand is the discounted expected cost of owning the option.

2.1.1. *Upper and lower price bounds.* In analogy with vanilla call options, the upper bound for the price of a best-of call is given by the maximum of the current underlying prices. The lower bound is given by the difference between the maximum underlying price and the discounted strike price. Arbitrage considerations and our numerical computations confirm these bounds.

2.2. **Pricing "best-of" rainbow options with stochastic correlations.** A dynamic process recently introduced in the literature to randomize both volatilities and correlations in a multi-asset framework is the multivariate Ornstein-Uhlenbeck process driven by a Lévy subordinator (Barndorff-Nielsen and Stelzer 2006; Pigorsch and Stelzer 2006). In this case, stochastic correlations may be accounted for by taking the subordinator to be a compound Poisson process with Wishart jumps (Stelzer 2006). The idea would be to derive a multivariate extension of the Barndorff-Nielsen and Shephard (henceforth BN-S 2001) stochastic volatility model and applied to derivative pricing by Nicolato and Venardos (2001). However, no study to date has proved the applicability of the model to price multi-asset derivatives in closed form.

[3] As there is no closed-form solution for the multivariate standard normal cdf, a numerical approximation is called upon; statistical software such as Matlab incorporate such approximations. When pricing derivatives with n underlyings, the n-variate standard normal cdf needs to be approximated and the approximation will become less exact as n increases, hence hindering the quality of the corresponding pricing formulas. For a discussion on the use of such approximations in derivative pricing, see Haug (2006), Ouwehand and West (2006), and West (2005).

2.2.1. *Multi-dimensional Ornstein-Uhlenbeck covariance matrix.* In what follows, S_t has the same dynamics as in 2.1. It will be useful to apply Ito's lemma and derive the dynamics of

$$\mathrm{d}\log S_t = (r\mathbb{I}_3 + \Sigma_t\beta)\,\mathrm{d}t + \sqrt{\Sigma_t}\mathrm{d}W_t^S, \tag{2.3}$$

with the same notation as above. The variance-covariance matrix is no longer constant, but follows a multivariate Ornstein-Uhlenbeck process driven by a matrix subordinator L_t:

$$\mathrm{d}\Sigma_t = \mathbf{A}\Sigma_t\mathrm{d}t + \mathrm{d}\tilde{W}_t. \tag{2.4}$$

The solution to the SDE is given by

$$\Sigma_t = \mathrm{e}^{\mathbf{A}t}\Sigma_0 + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathrm{d}\tilde{W}_s.$$

We take $\tilde{W}_t$ as a matrix Brownian motion, whereas Barndorff-Nielsen and Stelzer (2006) take a matrix subordinator L_t defined as a compound Poisson process with intensity λ (see Barndorff-Nielsen and Perez-Abreu 2006). To allow for stochastic correlations the subordinator may be a compound Poisson process with Wishart jumps. The solution to

$$\mathrm{d}L_t = \gamma\mathrm{d}t + \int_{\mathbb{S}_3^+\ \{0\}} x\mu(\mathrm{d}t, \mathrm{d}x)$$

is

$$L_t = \gamma t + \int_0^t \int_{\mathbb{S}_3^+\ \{0\}} x\mu(\mathrm{d}s, \mathrm{d}x)$$

where $\mu \in \mathbb{S}_3^+$ is a deterministic drift, $\mathbb{S}_3^+$ denotes symmetric positive semidefinite 3×3 matrices, and $\mu(\mathrm{d}s, \mathrm{d}x)$ an extended Poisson random measure.

We have

$$\begin{aligned}\log S_T &= \log S_t + \int_t^T (r + \beta\Sigma_s)\,\mathrm{d}s + \int_t^T \sqrt{\Sigma_t}\mathrm{d}W_s^S \\ &= \log S_t + r(T-t) + \beta\Sigma_{t,T}^* + \int_t^T \sqrt{\Sigma_t}\mathrm{d}W_s,\end{aligned}$$

where $\Sigma_{t,T}^*$ is the integrated volatility, i.e.

$$\begin{aligned}\Sigma_{t,T}^* &= \int_t^T \Sigma_s\mathrm{d}s = -\mathbf{A}^{-1}\left(1 - \mathrm{e}^{\mathbf{A}(T-t)}\right)\Sigma_t - \mathbf{A}^{-1}\int_t^T \left(1 - \mathrm{e}^{\mathbf{A}(T-s)}\right)\mathrm{d}L_s \\ &= \Upsilon(t,T)\Sigma_t + \int_t^T \Upsilon(s,T)\mathrm{d}\tilde{W}_s\end{aligned}$$

With matrix $\mathbf{A}$ chosen in such a way to preserve positive-semidefiniteness: The matrix-analogue is solution to the SDE

$$\mathrm{d}\Sigma_t = (A\Sigma_t + \Sigma_t A')\mathrm{d}t + \mathrm{d}\tilde{W}_t$$

and is given by

$$\Sigma_t = \mathrm{e}^{At}\Sigma_0\mathrm{e}^{A't} + \int_0^t \mathrm{e}^{A(t-s)}\mathrm{d}W_s\mathrm{e}^{A'(t-s)}.$$

Note that with $n = 1$ the system of equations (**??**) and (2.4) simplifies to:

$$\begin{aligned}\mathrm{d}\log S_t &= \left(r + \beta\sigma_t^2\right)\mathrm{d}t + \sigma_t\mathrm{d}W_t, \\ \mathrm{d}\sigma^2 &= -\lambda\sigma_t^2\mathrm{d}t + \mathrm{d}\tilde{W}_t,\end{aligned}$$

which is the Barndorff-Nielsen and Shephard (2001) model applied to derivative pricing by Nicolato and Venardos (2003).

Laplace transform of log-prices (with pricing at $t = 0$):

$$\begin{aligned}\Psi(z) &= \mathbb{E}\left[\exp\{z \log S_T\}\right] \\ &= \mathbb{E}\left[\exp\{z'\left(logS_0 + \mu T + \beta'\Sigma_0 + \int_0^T \Sigma_3\right)\}\right] \\ &=\end{aligned}$$

In the following, we assume $\tilde{W}$ and A diagonal (so $A = A'$); therefore,

$$\begin{aligned}\Psi(z) &= \mathbb{E}\left[\exp\{z\alpha + z\int \sqrt{\Sigma}\mathrm{d}W\}\right] \\ &= \mathbb{E}\left[\exp\{z\alpha\}\right]\mathbb{E}\left[\exp\{z\int \sqrt{\Sigma}\mathrm{d}W\}\right],\end{aligned}$$

where

$$\begin{aligned}\int_t^T \sqrt{\Sigma}\mathrm{d}W &= \int_t^T \sqrt{\int_{-\infty}^t \mathrm{e}^{A(t-s)}\mathrm{d}\tilde{W}\mathrm{e}^{A(t-s)}}\mathrm{d}W \\ &= \int_t^T \sqrt{\int_{-\infty}^t \mathrm{e}^{2A(t-s)}\mathrm{d}\tilde{W}_s}\mathrm{d}W \\ &= \int_t^T \sqrt{\int_{-\infty}^t \mathcal{A}_s\mathrm{d}\tilde{W}_s}\mathrm{d}W \\ &= \begin{pmatrix} \int_t^T \sqrt{\int_{-\infty}^t \mathcal{A}_s^{11}\mathrm{d}\tilde{W}_s^{11}}\mathrm{d}W & 0 & 0 \\ 0 & \int_t^T \sqrt{\int_{-\infty}^t \mathcal{A}_s^{22}\mathrm{d}\tilde{W}_s^{22}}\mathrm{d}W & 0 \\ 0 & 0 & \int_t^T \sqrt{\int_{-\infty}^t \mathcal{A}_s^{33}\mathrm{d}\tilde{W}_s^{33}}\mathrm{d}W \end{pmatrix}\end{aligned}$$

[. . . to follow . . .]

Baker-Campbell-Hausdorff formulae. Given a matrix A, define:

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Also, given another matrix B:

$$B_t = \mathrm{e}^{tA} B_0 \mathrm{e}^{-tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k;$$

when $t = 1$, then

$$\mathrm{e}^A B_0 \mathrm{e}^{-A} = \sum_{k=0}^{\infty} \frac{1}{k!} B_k.$$

Formula for the general case in which the two matrices do not commute:

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^Z,$$

where Z is a series, whose first terms are

$$\begin{aligned} z_1 &= A+B \\ z_2 &= \frac{1}{2}(AB-BA) \\ z_3 &= \frac{1}{12}(A^2B+AB^2-2ABA+B^2A+BA^2-2BAB) \end{aligned}$$

In more specific cases, if $[A,[A,B]]=[B,[B,A]]$, then $\mathrm{e}^A\mathrm{e}^B=\mathrm{e}^{A+B}\mathrm{e}^{\frac{1}{2}[A,B]}$; and if also $AB=BA$, then $\mathrm{e}^A\mathrm{e}^B=\mathrm{e}^{A+B}$. (Note: $[A,B]=AB-BA$, hence $[A,[A,B]]=A^2B+AB^2-2ABA$.)

2.2.2. *Superposition of one-dimensional Ornstein-Uhlenbeck processes.* Alternatively, one may model the covariance matrix from a set of n idiosyncratic and k common factors. The covariance matrix is then specified as

$$\Sigma_t = I_t + AJ_tA',$$

where

$$\begin{aligned} I_t &= \mathrm{diag}(\varphi_{1,t},\ldots,\varphi_{n,t}) \\ J_t &= \mathrm{diag}(\varphi_{n+1,t},\ldots,\varphi_{n+k,t}) \\ A &\in \mathbb{R}^{n\times k} \text{ is the factor loadings matrix} \end{aligned}$$

Each factor follows a (one-dimensional) Ornstein-Uhlenbeck process:

$$\mathrm{d}\varphi_{i,t} = -\lambda\varphi_{i,t}\mathrm{d}t + \mathrm{d}\tilde{W}_t$$

[...to follow ...]

3. Sensitivity analysis

Following the pricing functions discussed in the previous section, we perform a sensitivity analysis in order to check how option prices change as parameters or variables change.[4] We are especially interested in correlation vegas. This will be of fundamental use when determining appropriate hedging strategies in Section 5.

3.1. **Option sensitivities with constant correlations.** The greeks we consider for the constant correlation model are the deltas,

$$\begin{aligned} \frac{\partial C}{\partial S_i} &= \mathcal{N}_3(d_{iK},d^\flat_{ij},d^\flat_{ik}) + \mathcal{N}_3'(d_{iK},d^\flat_{ij},d^\flat_{ik})\left[\frac{1}{\sigma_i\sqrt{T}}+\frac{1}{\sigma_{ij}\sqrt{T}}+\frac{1}{\sigma_{ik}\sqrt{T}}\right] \\ &\quad -\mathcal{N}_3'(d_{jK},d^\flat_{ji},d^\flat_{jk})\frac{S_j}{S_i\sigma_{ij}\sqrt{T}} - \mathcal{N}_3'(d_{kK},d^\flat_{ki},d^\flat_{kj})\frac{S_k}{S_i\sigma_{ik}\sqrt{T}} \\ &\quad -K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,iK},d_{2,jK},d_{2,kK})\frac{1}{S_i\sigma_i\sqrt{T}}, \end{aligned} \tag{3.1}$$

[4] According to the model at hand, volatilities and correlations will be defined either as parameters or variables.

the gammas,

$$
\begin{aligned}
\frac{\partial^2 C}{\partial S_i^2} &= \mathcal{N}_3'(d_{iK}, d_{ij}^\flat, d_{ik}^\flat)\left[\frac{1}{S_i\sigma_i\sqrt{T}} + \frac{1}{S_i\sigma_{ij}\sqrt{T}} + \frac{1}{S_i\sigma_{ik}\sqrt{T}}\right] \\
&+\mathcal{N}_3''(d_{iK}, d_{ij}^\flat, d_{ik}^\flat)\left[\frac{1}{S_i\sigma_i\sqrt{T}} + \frac{1}{S_i\sigma_{ij}\sqrt{T}} + \frac{1}{S_i\sigma_{ik}\sqrt{T}}\right]\left[\frac{1}{\sigma_i\sqrt{T}} + \frac{1}{\sigma_{ij}\sqrt{T}} + \frac{1}{\sigma_{ik}\sqrt{T}}\right] \\
&+\mathcal{N}_3''(d_{jK}, d_{ji}^\flat, d_{jk}^\flat)\frac{S_j}{S_i^2\sigma_{ij}^2 T} + \mathcal{N}_3'(d_{jK}, d_{ji}^\flat, d_{jk}^\flat)\frac{S_j}{S_i^2\sigma_{ij}\sqrt{T}} \\
&+\mathcal{N}_3''(d_{kK}, d_{ki}^\flat, d_{kj}^\flat)\frac{S_k}{S_i^2\sigma_{ik}^2 T} + \mathcal{N}_3'(d_{kK}, d_{ki}^\flat, d_{kj}^\flat)\frac{S_k}{S_i^2\sigma_{ik}\sqrt{T}} \\
&-K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,iK}, d_{2,jK}, d_{2,kK})\frac{1}{S_i^2\sigma_i^2 T} \\
&+K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,iK}, d_{2,jK}, d_{2,kK})\frac{1}{S_i^2\sigma_i\sqrt{T}},
\end{aligned} \tag{3.2}
$$

the cross-gammas,

$$
\begin{aligned}
\frac{\partial^2 C}{\partial S_i \partial S_j} &= -\mathcal{N}_3'(d_{iK}, d_{ij}^\flat, d_{ik}^\flat)\frac{1}{S_j\sigma_{ij}\sqrt{T}} \\
&-\mathcal{N}_3''(d_{iK}, d_{ij}^\flat, d_{ik}^\flat)\frac{1}{S_j\sigma_{ij}\sqrt{T}}\left[\frac{1}{\sigma_i\sqrt{T}} + \frac{1}{\sigma_{ij}\sqrt{T}} + \frac{1}{\sigma_{ik}\sqrt{T}}\right] \\
&-\mathcal{N}_3''(d_{jK}, d_{ji}^\flat, d_{jk}^\flat)\left[\frac{1}{\sigma_j\sqrt{T}} + \frac{1}{\sigma_{ij}\sqrt{T}} + \frac{1}{\sigma_{jk}\sqrt{T}}\right]\frac{1}{S_i\sigma_{ij}\sqrt{T}} \\
&-\mathcal{N}_3'(d_{jK}, d_{ji}^\flat, d_{jk}^\flat)\frac{1}{S_i\sigma_{ij}\sqrt{T}} \\
&+\mathcal{N}_3''(d_{kK}, d_{ki}^\flat, d_{kj}^\flat)\frac{1}{S_j\sigma_{jk}\sqrt{T}}\frac{S_k}{S_i\sigma_{ik}\sqrt{T}} \\
&-K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,iK}, d_{2,jK}, d_{2,kK})\frac{1}{S_j\sigma_j\sqrt{T}}\frac{1}{S_i\sigma_i\sqrt{T}},
\end{aligned} \tag{3.3}
$$

the vegas,

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma_i} &= S_i\,\mathcal{N}_3'(d_{iK}, d_{ij}^\flat, d_{ik}^\flat)\frac{\sqrt{T}}{2}\left(1 + \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}} + \frac{\sigma_1 - \rho_{ik}\sigma_k}{\sigma_{ik}}\right) \\
&+S_j\,\mathcal{N}_3'(d_{jK}, d_{ji}^\flat, d_{jk}^\flat)\frac{\sqrt{T}}{2}\,\frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}} + S_k\,\mathcal{N}_3'(d_{kK}, d_{ki}^\flat, d_{kj}^\flat)\frac{\sqrt{T}}{2}\,\frac{\sigma_i - \rho_{ik}\sigma_k}{\sigma_{ik}} \\
&+K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,iK}, d_{2,jK}, d_{2,kK})\frac{\sqrt{T}}{2},
\end{aligned} \tag{3.4}
$$

and the correlation vegas,

$$
\begin{aligned}
\frac{\partial C}{\partial \rho_{ij}} &= S_i\,\mathcal{N}_3'(d_{iK}, d_{i\,i+1}^\flat, d_{i\,i+2}^\flat)\sqrt{T}\left(\sigma_j - \frac{\sigma_i\sigma_j}{\sigma_{ij}}\right) \\
&+S_j\,\mathcal{N}_3'(d_{jK}, d_{j\,j+1}^\flat, d_{j\,j+2}^\flat)\sqrt{T}\left(\sigma_i - \frac{\sigma_i\sigma_j}{\sigma_{ij}}\right) \\
&-K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,iK}, d_{2,jK}, d_{2,kK})\sqrt{T}(\sigma_i + \sigma_j).
\end{aligned} \tag{3.5}
$$

Appendix B illustrates detailed calculations for each one of these sensitivities. Note that the greeks above make use of the triviate standard normal probability function,which is given by

$$\frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}}\int_{-\infty}^{d_1}\int_{-\infty}^{d_2}\int_{-\infty}^{d_3}\exp\{\frac{1}{2}\underline{x}'\Sigma^{-1}\underline{x}\}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

with

$$\Sigma=\begin{pmatrix}1 & \rho_{12} & \rho_{13}\\ \rho_{12} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{pmatrix}.$$

3.1.1. *Deltas.* Equation (3.1) estimates the sensitivity of the option price to changes in the price of the ith underlying, ie the delta wrt the ith underlying. This gives a 3-component delta vector. In this case, we cannot simply recur to Euler's homogeneous function theorem and state that $\frac{\partial C}{\partial S_i}=\mathcal{N}_3(d_{iK},d^{\flat}_{ij},d^{\flat}_{ik})$. If we did, we would subsequently get incorrect calculations for the gammas and cross-gammas, since the equivalence $\frac{\partial^2 C}{\partial S_1\partial S_2}=\frac{\partial^2 C}{\partial S_2\partial S_1}$ would not subsist.[5]

For pure illustrative reasons, we may compare this delta to the usual formula obtained within a single-asset Black-Scholes framework: in the present case, (3.6) shows that one gets further components based on the relative performance of asset i compared to the other underlying assets. These additional terms appear with a minus sign, since the presence of additional underlyings makes the impact of one specific underlying on the option price less dominant:

$$\Delta_i=\frac{\partial C}{\partial S_i}=[\text{Black-Scholes}]_i-\sum_{j=1}^{n}\mathcal{N}_3'(d_{jK},d^{\flat}_{j\,j+1},d^{\flat}_{j\,j+2})\frac{S_j}{S_i\sigma_{ij}\sqrt{T}}. \tag{3.6}$$

Furthermore, suppose a (long) call option is deep in-the-money in S_1; then Δ_1 becomes larger, whereas Δ_2 and Δ_3 are very close to zero. These deltas get even more polarized as the volatility of S_1 decreases or as the option approaches maturity.

3.1.2. *Gammas and cross-gammas.* Equation (3.2) estimates the sensitivity of the delta wrt the ith underlying to changes in the price of the same underlying, ie the (individual) gamma wrt the ith underlying. Equation (3.3) estimates the sensitivity of the delta wrt the ith underlying to changes in the price of the jth underlying, ie the cross-gamma wrt the ith and jth underlyings. For a rainbow option with three underlyings, we need to compute three individual gammas and three cross-gammas.[6] This gives a 3×3 gamma matrix.

It is interesting to note that a long position in a best-of call has positive gammas but negative cross-gammas. Intuitively, these measures are relevant for at-the-money options: as S_i increases, Δ_i increases but Δ_j decreases. We may interpret cross-gammas as a measure of the stability of Δ_j as other underlyings' prices increase.

Nevertheless, we believe cross-gammas are more interesting to look at when hedging, say, spread options, rather than best-of or worst-of options. For the latter type of options, only the best- or worst-performing underlying affects the payoff,

[5]According to Euler's homogeneous function theorem, if $f(\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4)=\lambda f(x_1,x_2,x_3,x_4)$ for any constant λ, then $f(x_1,x_2,x_3,x_4)=x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+x_3\frac{\partial f}{\partial x_3}+x_4\frac{\partial f}{\partial x_4}$.

[6]More generally, a rainbow option with n underlyings would have $n(n+1)/2$ gammas given Schwartz theorem that $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$.

regardless of any difference between the underlyings' movements; in our opinion, this makes cross-gammas less meaningful.

3.1.3. *Vegas.* Equation (3.4) estimates the sensitivity of the option price to changes in the volatility of the ith underlying, ie the vega wrt the ith underlying. This gives a 3-component vega vector. A long position in a best-of call has positive vegas, since higher price volatility for any one of the underlyings means a greater probability for the option to expire in-the-money.

Here again, we may compare (3.4) with the single-asset Black-Scholes vega, which leads us to equation (3.7):

$$\frac{\partial C}{\partial \sigma_i} = [\text{Black-Scholes}]_i + \sum_{j=1}^{n} S_j \, \mathcal{N}_3'(d_{jK}, d^b_{j\,j+1}, d^b_{j\,j+2}) \sqrt{T} \, \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}}. \tag{3.7}$$

Although the additional terms in (3.7) appear with a plus sign, we cannot state that we get higher vegas than the usual Black-Scholes framework, since the signs of these terms depend on the correlation coefficients. At first sight, it would seem that the vegas wrt the ith and jth underlyings are positive when the two underlyings are negatively correlated; however, it is in reality very difficult to disentangle the effects of the volatilities from those of the correlations.

3.1.4. *Correlation vegas.* Equation (3.5) estimates the sensitivity of the option price to changes in the correlation between the ith and jth assets, ie the correlation vega wrt the ith and jth underlyings. This gives a 3×3 triangular correlation vega matrix. As discussed above, best-of rainbow options are more valuable the lower the correlations among their underlyings; hence, we expect to derive negative correlation vegas. However, the sign of this greek is not straightforward to determine and being long or short correlation appears to depend on the intial values of the correlation coefficients. Negative correlation vegas would be guaranteed when

$$\left(\sigma_j - \frac{\sigma_i \sigma_j}{\sigma_{ij}}\right) < 0 \quad \text{and} \quad \left(\sigma_i - \frac{\sigma_i \sigma_j}{\sigma_{ij}}\right) < 0;$$

but these conditions are never satisfied contemporaneously. Given the large number of parameters in (3.5), numerical computations are of greater help in understanding this greek's direction. Our numerical computations show that correlation vegas vary widely for different starting correlation coefficients, ρ_{ij}, ρ_{ik}, and ρ_{jk}. A couple of examples are presented in Figures 1 and 2 below. Figure 1 shows that when all three correlations are positive, correlation vega is negative; hence, an increase in ρ_{12} leads to a decrease in option price. Different starting values of ρ_{12} lead to different correlation vegas. Figure 2 shows that when ρ_{12} is negative, but the other ρ's are positive, correlation vega turns positive for higher values of ρ_{23}. The dependence of $\frac{\partial C}{\partial \rho_{12}}$ on not only ρ_{ij} but also ρ_{ik} and ρ_{jk} will be important when identifying appropriate hedging strategies.

3.1.5. *Other sensitivities.* We may also calculate the volga matrix (or DVegaDVol) of a best-of call option. The holder of this option would be long both volgas and cross-volgas: greater dispersion among underlyings' movements translates into a higher probability of the option expiring in-the-money in one of its underlyings. Moreover, our numerical computations show that a long position in the best-of call has negative theta, so that the option becomes less valuable as time decays, and

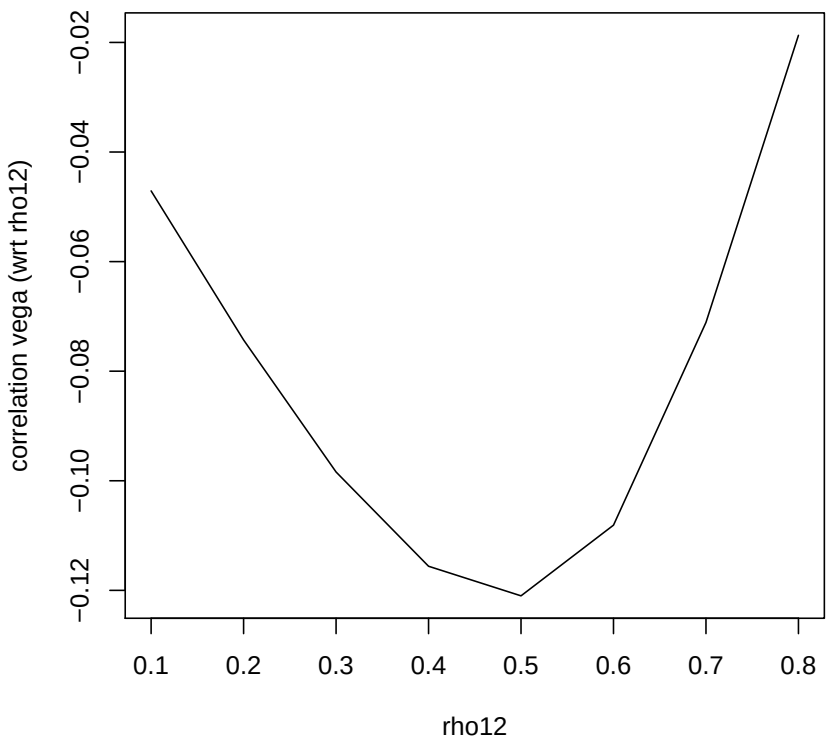

FIGURE 1. Variation of correlation vega $\frac{\partial C}{\partial \rho_{12}}$ for different initial values for ρ_{12}. Other starting values are $S_1 = 100$, $S_2 = 120$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $\sigma_3 = 0.3$, $r = 0.05$, $T = 1$, $\rho_{13} = 0.3$ and $\rho_{23} = 0.6$.

has a positive rho, so that an increase in interest rates makes it more favorable to hold the option than to buy the stock.

3.2. **Option sensitivities with stochastic correlations.** [...to follow ...]

Note that the pricing formula of the rainbow option with the Wishart volatility dynamics involves a multidimensional complex integral. It has been suggested to me to try and compute the greeks via Malliavin Calculus...I will try and give it a month, then -if unsuccessful- I will finally opt for a numerical method...

4. Market calibration

[...to follow ...]

4.0.1. *Data.* In this paper we make use of data from Bloomberg ranging from...to...We take prices of rainbow options on three underlyings, together with time series of the underlyings' prices.

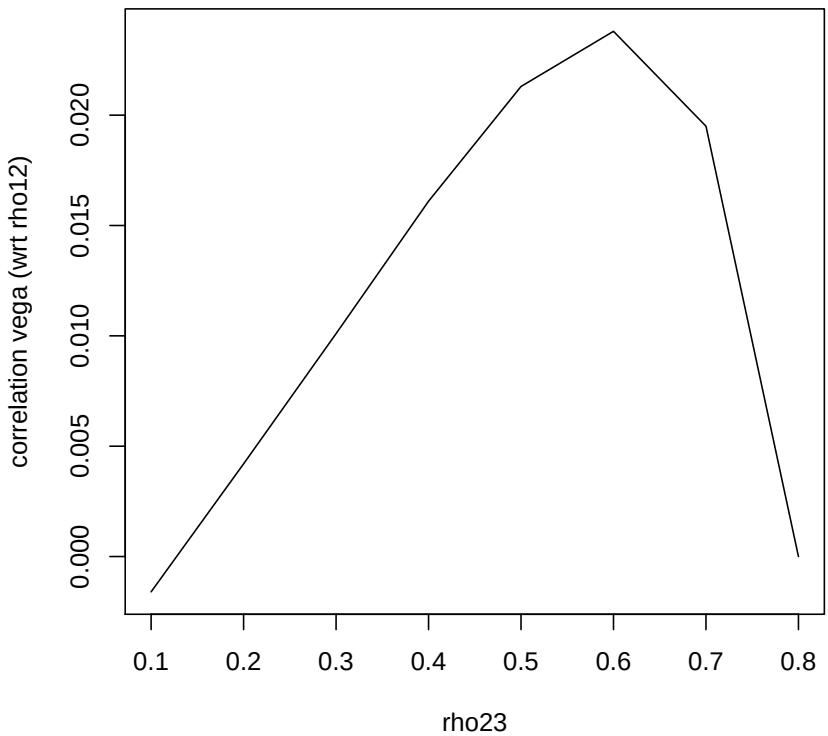

FIGURE 2. Variation of correlation vega $\frac{\partial C}{\partial \rho_{12}}$ for different initial values for ρ_{23}. Other starting values are $S_1 = 100$, $S_2 = 120$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $\sigma_3 = 0.3$, $r = 0.05$, $T = 1$, $\rho_{12} = -0.3$ and $\rho_{13} = 0.3$.

4.0.2. *Methodology.* Estimate model parameters by minimizing the difference between market prices and model prices in a least-squares sense. Compute several global measures of fit:
Root mean square error:

$$rmse = \sqrt{\sum \frac{(\text{market price} \ - \ \text{model price})^2}{\text{number of options}}}$$

Average (absolute) percentage error:

$$ape = \frac{1}{\text{mean option price}} \sum \frac{|\text{market price} \ - \ \text{model price}|}{\text{number of options}}$$

Average absolute error:

$$aae = \sum \frac{|\text{market price} \ - \ \text{model price}|}{\text{number of options}}$$

Average relative percentage error:

$$arpe = \frac{1}{\text{number of options}} \sum \frac{|\text{market price} \ - \ \text{model price}|}{\text{number of options}}$$

4.1. **Model calibration with constant correlations.** [. . . to follow . . .]

4.2. **Model calibration with stochastic correlations.** [. . . to follow . . .]

5. Hedging best-of call options

Correlation risk is of interest not only for option pricers, but also for traders who need to hedge their positions in multi-asset derivatives. The aim of this section is to find the optimal hedging strategies from the perspective of those holding rainbow options. They face the trade-off between the insurance obtained through hedging and the costs incurred by such insurance and the questions they face are which particular sensitivities to minimize and at what frequency to hedge.

5.1. **Hedging with constant correlations.** The greeks discussed throughout Section 3 guide us through our choice of ways to hedge a three-color best-of call option. The complexity encountered when hedging rainbow options with constant correlations should serve as a preview of further complexities to be dealt with when including stochastic correlations![7]

5.1.1. *First- and second-order underlying asset risks.* The first-order underlying risks are represented by the delta vector. As all three components of this vector are of positive sign, the first-order underlying risks can be minimized by selling shares of each underlying by an amount equal to the corresponding delta. The second-order underlying risks are, instead, represented by the gamma matrix and are more complex to deal with. The diagonal elements of the gamma matrix can be neutralized with stock and vanilla options, whereas the non-diagonal elements (i.e. the cross-gammas) require the use of other multi-asset options. Note that adding to one's position other multi-asset options will affect position correlation vega, which would then require additional hedging instruments if it is, too, to be minimized.

5.1.2. *First- and second-order volatility risks.* The holder of best-of call is long vega, so that the first-order volatility risk can be minimized by selling(buying) traded options with positive(negative) vegas. Here again, second-order risks require a more involved strategy. The diagonal elements of the volga matrix with stock and vanillas; non-diagonal elements (ie cross-volgas) with other multi-asset options. As in practice not all risk exposures can be addressed because of transaction costs and other practical issues, we may leave volga risk aside without loss of . . . .

[7] As multi-asset options may differ substantially one from the other, no one-size-fits-all hedging strategy can apply. For hedging issues related to multi-asset options other than rainbows, see, eg, Overhaus (2002) for Himalayas and Fengler and Schwendner (2006) for baskets.

5.1.3. *First-order correlation risk.* Exposure to correlation risk is rather cumbersome to deal with, since position correlation vega changes as cross-gammas (and eventually cross-volgas) are neutralized through the use of multi-asset traded options. It is also quite difficult to disentangle correlations from volatilities, as illustrated in the previous section. Thirdly, an annoying factor remains that correlations are not directly observable![8]

5.1.4. *An exemplifying hedging strategy.* The aim of this subsection is to draw light on the intricacies associated with hedging correlation-based products. We outline a possible strategy based on the greeks we have computed and discussed so far. In this example, we consider a delta-gamma-vega-correlation vega-neutral strategy.

If all underlyings belong to the same index, the investor may consider hedging her correlation risk by selling or buying an index option. The index option is typically less volatile than options written on the individual components; however, the investor will expose herself to new macroeconomic risks. Alternatively, correlation swaps have allowed banks in the last years to hedge part of their correlation positions. These are forward contracts on the realized correlation of bespoke baskets (rather than indices).

An exemplifying hedging strategy for a three-color rainbow option makes use of a total of fifteen instruments: three stocks, six vanilla options, and six multi-asset options. It is easy to see that as the number of underlyings increases, the ideal number of hedging instruments grows immensely. Furthermore, up to this point, we have totally ignored any costs associated with the transactions: these costs can become burdensome enough to hinder appropriate hedging of the investor's position.

[. . . to follow . . .]

5.2. **Hedging with stochastic correlations.** [. . . to follow . . .]

6. Conclusions

[. . . to follow . . .]

[8] We thank Umberto Cherubini for pointing out an interesting rule-of-thumb to check whether the buyer(seller) of a multi-asset option is long or short correlation: the buyer(seller) is long(short) correlation if the option's payoff is defined on the "and" operator; vice versa, if the options's payoff is defined on the "or" operator. Nevertheless, we believe that the dependence of correlation vegas on initial correlation coefficients does not always allow for such a simple rule-of-thumb.

References

Barndorff-Nielsen, C. and R. Perez-Abreu (2007), "Matrix subordinators and upsilon transformations", *Theoretical Probability*, forthcoming.

Barndorff-Nielsen, C. and N. Shephard (2001), "Non-gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics", *Journal of the Royal Statistics Society*, 63(2), pp. 167-241.

Barndorff-Nielsen, C. and R. Stelzer (2006), "Positive-definite matrix processes of finite variation", Working paper.

Bru, M. F. (1991), "Wishart processes", *Journal of Theoretical Probability*, 4(4), pp. 725-751.

Carr, P. and D. B. Madan (1999), "Option pricing using the fast Fourier transform", *Journal of Computational Finance*, 2(4), pp. 61-73.

Fengler, M. R. and P. Schwendner (2003), "Correlation risk premia for multi-asset equity derivatives", Working paper.

Hubalek, F. and E. Nicolato (2007), "Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type", Working paper.

Haug, E. G. (2007), *Option pricing formulas*, Wiley Series in Financial Engineering.

Jeffrey, C. (2004), "Shedding the correlation 'axe' ", *Risk Magazine*, 17(5), pp. ???-???.

Johnson, H. (1987), "Call on the maximum or minimum of several assets", *Journal of Financial and Quantitative Analysis*, 22(3), pp. 277-283.

Nicolato, E. and E. Venardos (2003), "Option pricing in Ornstein-Uhlenbeck type stochastic volatility models", *Mathematical Finance*, 13(4), pp. 445-466.

Overhaus, M. (2002), "Himalaya options", *Risk Magazine*, 15(3), pp. 101-104.

Pigorsch, C. and R. Stelzer (2007), "A multivariate generalization of the Ornstein-Uhlenbeck stochastic volatility model", Working paper.

Rubinstein, H. (1995), "Somewhere over the rainbow", *Risk Magazine*, 8(11), pp. 63-66.

Stelzer, R. (2007), personal e-mail communication, August 3.

Stulz, R. M. (1982), "Options on the minimum or the maximum of two risky assets - Analysis and applications", *Journal of Financial Economics*, 10(2), pp. 161-185.

Taleb, N. N. (1997), *Dynamic Hedging: Managing Vanilla and Exotic Options*, Wiley Series in Financial Engineering.

Ouwehand, P. and G. West (2006), "Pricing rainbow options", Working paper.

West, G. (2005), "Better approximations to cumulative normal functions", *Wilmott Magazine*, 5, pp. 70-76.

Wilmott, P. (2006), *Frequently Asked Questions in Quantitative Finance*, Wiley Series in Finance.

Wystup, U. (2002), "How the greeks would have hedged correlation risk", in Hakala, J. and U. Wystup (ed.s), *Foreign Exchange Risk - Models, Instruments, and Strategies*, RiskBooks.

Appendix A. Sensitivities

Let C denote the price of the three-color best-of call option:

$$\begin{aligned} C &= S_1 \mathcal{N}_3\left(d_1(S_1,K,\sigma_1^2), d_1^\flat(S_1,S_2,\sigma_{12}^2), d_1^\flat(S_1,S_3,\sigma_{13}^2), \rho_{112},\rho_{113},\rho_{123}\right) \\ &\quad + S_2 \mathcal{N}_3\left(d_1(S_2,K,\sigma_2^2), d_1^\flat(S_2,S_1,\sigma_{12}^2), d_1^\flat(S_2,S_3,\sigma_{23}^2), \rho_{212},\rho_{213},\rho_{223}\right) \\ &\quad + S_3 \mathcal{N}_3\left(d_1(S_3,K,\sigma_3^2), d_1^\flat(S_3,S_1,\sigma_{13}^2), d_1^\flat(S_3,S_2,\sigma_{23}^2), \rho_{312},\rho_{313},\rho_{323}\right) \\ &\quad - K\mathrm{e}^{-rT}\left[1-\mathcal{N}_3\left(-d_2(S_1,K,\sigma_1^2), -d_2(S_2,K,\sigma_2^2), -d_2(S_3,K,\sigma_3^2), \rho_{12},\rho_{13},\rho_{23}\right)\right], \end{aligned}$$

with

$$d_1(S_i,K,\sigma_i^2) = \frac{\log\frac{S_i}{K} + \left(r+\frac{\sigma_i^2}{2}\right)T}{\sigma_i\sqrt{T}},$$

$$d_2(S_i,K,\sigma_i^2) = \frac{\log\frac{S_i}{K} + \left(r-\frac{\sigma_i^2}{2}\right)T}{\sigma_i\sqrt{T}},$$

$$d_1^\flat(S_i,S_j,\sigma_{ij}^2) = \frac{\log\frac{S_i}{S_j} + \frac{\sigma_{ij}^2}{2}T}{\sigma_{ij}\sqrt{T}},$$

and

$$\sigma_{ij}^2 = \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2.$$

A.1. **Deltas.** We may first rewrite the price of the option as the following cyclical sum (so that $s_1, s_2, s_3, s_4 = s_1, s_5 = s_2, \ldots$):

$$\begin{aligned} C &= \sum_{j=1}^{3} S_j \mathcal{N}_3\left(d_1(S_j,K,\sigma_j^2), d_1^\flat(S_j,S_{j+1},\sigma_j^2), d_1^\flat(S_j,S_{j+2},\sigma_j^2), \rho_j\right) \\ &\quad - K\exp\{-rT\}\left[1-\mathcal{N}_3\left(-d_2(S_1,K,\sigma_1^2), -d_2(S_2,K,\sigma_2^2), -d_2(S_3,K,\sigma_3^2), \rho_{12},\rho_{13},\rho_{23}\right)\right] \end{aligned}$$

According to the chain rule, the derivative of $g = f(u(x,y), v(x,y))$ wrt x is given by

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

Application of the chain rule to the pricing formula gives

$$\begin{aligned} \frac{\partial C}{\partial S_1} &= \mathcal{N}_3 + S_1\left[\frac{\partial \mathcal{N}_3}{\partial d_1(S_1,K,\sigma_1^2)}\frac{\partial d_1(S_1,K,\sigma_1^2)}{\partial S_1} + \frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_1,S_2,\sigma_{12}^2)}\frac{\partial d_1^\flat(S_1,S_2,\sigma_{12}^2)}{\partial S_1}\right. \\ &\quad \left. + \frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_1,S_3,\sigma_{13}^2)}\frac{\partial d_1^\flat(S_1,S_3,\sigma_{13}^2)}{\partial S_1}\right] \\ &\quad + S_2\frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_2,S_1,\sigma_{12}^2)}\frac{\partial d_1^\flat(S_2,S_1,\sigma_{12}^2)}{\partial S_1} + S_3\frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_3,S_1,\sigma_{13}^2)}\frac{\partial d_1^\flat(S_3,S_1,\sigma_{13}^2)}{\partial S_1} \\ &\quad - K\mathrm{e}^{-rT}\frac{\partial \mathcal{N}_3}{\partial d_2(S_1,K,\sigma_1^2)}\frac{\partial d_2(S_1,K,\sigma_1^2)}{\partial S_1}. \end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial C}{\partial S_1} &= \mathcal{N}_3 + S_1\left[\frac{\partial \mathcal{N}_3}{\partial d_1(S_1,K,\sigma_1^2)}\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_1,S_2,\sigma_{12}^2)}\frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_1,S_3,\sigma_{13}^2)}\frac{1}{S_1\sigma_{13}\sqrt{T}}\right]\\ &\quad -S_2\frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_2,S_1,\sigma_{12}^2)}\frac{1}{S_1\sigma_{12}\sqrt{T}} - S_3\frac{\partial \mathcal{N}_3}{\partial d_1^\flat(S_3,S_1,\sigma_{13}^2)}\frac{1}{S_1\sigma_{13}\sqrt{T}}\\ &\quad -K\mathrm{e}^{-rT}\frac{\partial \mathcal{N}_3}{\partial d_2(S_1,K,\sigma_1^2)}\frac{1}{S_1\sigma_1\sqrt{T}}.\end{aligned}$$

We may simplify the notation with

$$d_{iK} \triangleq d_1(S_i,K,\sigma_i^2),$$

$$d_{2,iK} \triangleq d_2(S_i,K,\sigma_i^2),$$

$$d_{ij}^\flat \triangleq d_1^\flat(S_i,S_j,\sigma_{ij}^2),$$

and denote by $\mathcal{N}_3'(\underline{d})$ the total derivative of $\mathcal{N}_3(\underline{d})$ (i.e. $\mathcal{N}_3'$ is the multivariate normal pdf); therefore:

$$\begin{aligned}\frac{\partial C}{\partial S_1} &= \mathcal{N}_3(d_{1K},d_{12}^\flat,d_{13}^\flat) + S_1\mathcal{N}_3'(d_{1K},d_{12}^\flat,d_{13}^\flat)\left[\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{1}{S_1\sigma_{13}\sqrt{T}}\right]\\ &\quad -S_2\,\mathcal{N}_3'(d_{2K},d_{21}^\flat,d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}} - S_3\,\mathcal{N}_3'(d_{3K},d_{31}^\flat,d_{32}^\flat)\frac{1}{S_1\sigma_{13}\sqrt{T}}\\ &\quad -K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,1K},d_{2,2K},d_{2,3K})\frac{1}{S_1\sigma_1\sqrt{T}}\\ &= \mathcal{N}_3(d_{1K},d_{12}^\flat,d_{13}^\flat) + \mathcal{N}_3'(d_{1K},d_{12}^\flat,d_{13}^\flat)\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right]\\ &\quad -\mathcal{N}_3'(d_{2K},d_{21}^\flat,d_{23}^\flat)\frac{S_2}{S_1\sigma_{12}\sqrt{T}} - \mathcal{N}_3'(d_{3K},d_{31}^\flat,d_{32}^\flat)\frac{S_3}{S_1\sigma_{13}\sqrt{T}}\\ &\quad -K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,1K},d_{2,2K},d_{2,3K})\frac{1}{S_1\sigma_1\sqrt{T}},\end{aligned} \tag{A.1}$$

where

$$\mathcal{N}_3'(\underline{d}) = \frac{1}{(2\pi)^{\frac{3}{2}}\,|\Sigma|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}(d-\mu)'\Sigma^{-1}(d-\mu)\right\}.$$

Analogously, the delta wrt the second underlying is

$$\begin{aligned}\frac{\partial C}{\partial S_2} &= -\mathcal{N}_3'(d_{1K},d_{12}^\flat,d_{13}^\flat)\frac{S_1}{S_2\sigma_{12}\sqrt{T}}\\ &\quad +\mathcal{N}_3(d_{2K},d_{21}^\flat,d_{23}^\flat) + \mathcal{N}_3'(d_{2K},d_{21}^\flat,d_{23}^\flat)\left[\frac{1}{\sigma_2\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{23}\sqrt{T}}\right]\\ &\quad -\mathcal{N}_3'(d_{3K},d_{31}^\flat,d_{32}^\flat)\frac{S_3}{S_2\sigma_{23}\sqrt{T}}\\ &\quad -K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,1K},d_{2,2K},d_{2,3K})\frac{1}{S_2\sigma_2\sqrt{T}}.\end{aligned} \tag{A.2}$$

We believe this cannot be simplified by use of Euler's homogeneous function theorem, since we would otherwise get incorrect results for the gammas and cross-gammas.

We may compare these deltas with the usual delta obtained in the Black-Scholes single-underlying framework:

$$\begin{aligned}\frac{\partial C}{\partial S} &= \frac{\partial}{\partial S}[S\mathcal{N}\left(d_1(S,K)\right) - K\mathrm{e}^{-rT}\mathcal{N}\left(d_2(S,K)\right)] \\ &= \mathcal{N} + S\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}d_1}\frac{\partial d_1}{\partial S} - K\mathrm{e}^{-rT}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}d_2}\frac{\partial d_2}{\partial S} \\ &= \mathcal{N} + \frac{\mathrm{d}\mathcal{N}}{\mathrm{d}d_1}\frac{1}{\sigma\sqrt{T}} - \frac{K}{S\sigma\sqrt{T}}\mathrm{e}^{-rT}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}d_2}, \qquad \text{(A.3)}\end{aligned}$$

where it can be shown that the last two summands cancel out by applying Euler's homogeneous function theorem, so that $\Delta = \mathcal{N}(d_1)$.

Comparing (A.1) with (A.3), we see that the delta of stock 1 has two components: the first component is the usual result obtained in a Black-Scholes framework if only the first asset were included, while the second component is performance-related:

$$\frac{\partial C}{\partial S_1} = [\text{Black-Scholes}]_1 - \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{S_2}{S_1\sigma_{12}\sqrt{T}} - \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{S_3}{S_1\sigma_{13}\sqrt{T}}$$

Following the same procedure, it can be shown that

$$\frac{\partial C}{\partial S_2} = [\text{Black-Scholes}]_2 - \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{S_1}{S_2\sigma_{12}\sqrt{T}} - \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{S_3}{S_2\sigma_{23}\sqrt{T}}$$

$$\frac{\partial C}{\partial S_3} = [\text{Black-Scholes}]_3 - \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{S_1}{S_3\sigma_{13}\sqrt{T}} - \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{S_2}{S_3\sigma_{23}\sqrt{T}}$$

A.2. **Individual and cross-gammas.** Calculating the gamma wrt the first underlying from (A.1), we derive

$$
\begin{aligned}
\frac{\partial^2 C}{\partial S_1^2} &= \left(\frac{\partial \mathcal{N}_3}{\partial d_{1K}}\frac{\partial d_{1K}}{\partial S_1} + \frac{\partial \mathcal{N}_3}{\partial d_{12}^\flat}\frac{\partial d_{12}^\flat}{\partial S_1} + \frac{\partial \mathcal{N}_3}{\partial d_{13}^\flat}\frac{\partial d_{13}^\flat}{\partial S_1}\right) \\
&\quad + \left(\frac{\partial \mathcal{N}_3'}{\partial d_{1K}}\frac{\partial d_{1K}}{\partial S_1} + \frac{\partial \mathcal{N}_3'}{\partial d_{12}^\flat}\frac{\partial d_{12}^\flat}{\partial S_1} + \frac{\partial \mathcal{N}_3'}{\partial d_{13}^\flat}\frac{\partial d_{13}^\flat}{\partial S_1}\right)\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad - \frac{\partial \mathcal{N}_3'}{\partial d_{21}^\flat}\frac{\partial d_{21}^\flat}{\partial S_1}\frac{S_2}{S_1\sigma_{12}\sqrt{T}} + \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{S_2}{S_1^2\sigma_{12}\sqrt{T}} \\
&\quad - \frac{\partial \mathcal{N}_3'}{\partial d_{31}^\flat}\frac{\partial d_{31}^\flat}{\partial S_1}\frac{S_3}{S_1\sigma_{13}\sqrt{T}} + \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{S_3}{S_1^2\sigma_{13}\sqrt{T}} \\
&\quad - K\mathrm{e}^{-rT}\frac{\partial \mathcal{N}_3'}{\partial d_{2,1K}}\frac{\partial d_{2,1K}}{\partial S_1}\frac{1}{S_1\sigma_1\sqrt{T}} + K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1^2\sigma_1\sqrt{T}} \\
&= \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\left[\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{1}{S_1\sigma_{13}\sqrt{T}}\right] \\
&\quad + \mathcal{N}_3''(d_{1K}, d_{12}^\flat, d_{13}^\flat)\left[\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{1}{S_1\sigma_{13}\sqrt{T}}\right]\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad + \mathcal{N}_3''(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}}\frac{S_2}{S_1\sigma_{12}\sqrt{T}} + \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{S_2}{S_1^2\sigma_{12}\sqrt{T}} \\
&\quad + \mathcal{N}_3''(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{1}{S_1\sigma_{13}\sqrt{T}}\frac{S_3}{S_1\sigma_{13}\sqrt{T}} + \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{S_3}{S_1^2\sigma_{13}\sqrt{T}} \\
&\quad - K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1\sigma_1\sqrt{T}}\frac{1}{S_1\sigma_1\sqrt{T}} + K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1^2\sigma_1\sqrt{T}} \\
&= \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\left[\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{1}{S_1\sigma_{13}\sqrt{T}}\right] \\
&\quad + \mathcal{N}_3''(d_{1K}, d_{12}^\flat, d_{13}^\flat)\left[\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{1}{S_1\sigma_{13}\sqrt{T}}\right]\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad + \mathcal{N}_3''(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{S_2}{S_1^2\sigma_{12}^2 T} + \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{S_2}{S_1^2\sigma_{12}\sqrt{T}} \\
&\quad + \mathcal{N}_3''(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{S_3}{S_1^2\sigma_{13}^2 T} + \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{S_3}{S_1^2\sigma_{13}\sqrt{T}} \\
&\quad - K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1^2\sigma_1^2 T} + K\mathrm{e}^{-rT}\mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1^2\sigma_1\sqrt{T}}.
\end{aligned}
$$

Calculating cross-gamma from (A.1), we derive

$$
\begin{aligned}
\frac{\partial^2 C}{\partial S_1 \partial S_2} &= \frac{\partial \mathcal{N}_3}{\partial d_{12}^\flat}\frac{\partial d_{12}^\flat}{\partial S_2} + \frac{\partial \mathcal{N}_3'}{\partial d_{12}^\flat}\frac{\partial d_{12}^\flat}{\partial S_2}\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad - \left[\left(\frac{\partial \mathcal{N}_3'}{\partial d_{2K}}\frac{\partial d_{2K}}{\partial S_2} + \frac{\partial \mathcal{N}_3'}{\partial d_{21}^\flat}\frac{\partial d_{21}^\flat}{\partial S_2} + \frac{\partial \mathcal{N}_3'}{\partial d_{23}^\flat}\frac{\partial d_{23}^\flat}{\partial S_2}\right)\frac{S_2}{S_1\sigma_{12}\sqrt{T}} + \mathcal{N}_3'\frac{1}{S_1\sigma_{12}\sqrt{T}}\right] \\
&\quad - \frac{\partial \mathcal{N}_3'}{\partial d_{32}^\flat}\frac{\partial d_{32}^\flat}{\partial S_2}\frac{S_3}{S_1\sigma_{13}\sqrt{T}} - K\mathrm{e}^{-rT}\frac{\partial \mathcal{N}_3'}{\partial d_{2,2K}}\frac{\partial d_{2,2K}}{\partial S_2}\frac{1}{S_1\sigma_1\sqrt{T}} \\
&= -\mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{1}{S_2\sigma_{12}\sqrt{T}} \\
&\quad -\mathcal{N}_3''(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{1}{S_2\sigma_{12}\sqrt{T}}\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad - \left[\mathcal{N}_3''(d_{2K}, d_{21}^\flat, d_{23}^\flat)\left[\frac{1}{S_2\sigma_2\sqrt{T}} + \frac{1}{S_2\sigma_{12}\sqrt{T}} + \frac{1}{S_2\sigma_{23}\sqrt{T}}\right]\frac{S_2}{S_1\sigma_{12}\sqrt{T}}\right. \\
&\quad \left. + \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}}\right] \\
&\quad +\mathcal{N}_3''(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{1}{S_2\sigma_{23}\sqrt{T}}\frac{S_3}{S_1\sigma_{13}\sqrt{T}} \\
&\quad -K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_2\sigma_2\sqrt{T}}\frac{1}{S_1\sigma_1\sqrt{T}} \\
&= -\mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{1}{S_2\sigma_{12}\sqrt{T}} \\
&\quad -\mathcal{N}_3''(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{1}{S_2\sigma_{12}\sqrt{T}}\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad -\mathcal{N}_3''(d_{2K}, d_{21}^\flat, d_{23}^\flat)\left[\frac{1}{\sigma_2\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{23}\sqrt{T}}\right]\frac{1}{S_1\sigma_{12}\sqrt{T}} \\
&\quad -\mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}} \\
&\quad +\mathcal{N}_3''(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{1}{S_2\sigma_{23}\sqrt{T}}\frac{S_3}{S_1\sigma_{13}\sqrt{T}} \\
&\quad -K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_2\sigma_2\sqrt{T}}\frac{1}{S_1\sigma_1\sqrt{T}}.
\end{aligned}
$$

Calculating cross-gamma from (A.2), we derive

$$
\begin{aligned}
\frac{\partial^2 C}{\partial S_2 \partial S_1} &= -\left[\left(\frac{\partial \mathcal{N}_3'}{\partial d_{1K}}\frac{\partial d_{1K}}{\partial S_1} + \frac{\partial \mathcal{N}_3'}{\partial d_{12}^\flat}\frac{\partial d_{12}^\flat}{\partial S_1} + \frac{\partial \mathcal{N}_3'}{\partial d_{13}^\flat}\frac{\partial d_{13}^\flat}{\partial S_1}\right)\frac{S_1}{S_2\sigma_{12}\sqrt{T}} + \mathcal{N}_3'\frac{1}{S_2\sigma_{12}\sqrt{T}}\right] \\
&\quad + \frac{\partial \mathcal{N}_3}{\partial d_{21}^\flat}\frac{\partial d_{21}^\flat}{\partial S_1} + \frac{\partial \mathcal{N}_3'}{\partial d_{21}^\flat}\frac{\partial d_{21}^\flat}{\partial S_1}\left[\frac{1}{\sigma_2\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{23}\sqrt{T}}\right] \\
&\quad - \frac{\partial \mathcal{N}_3'}{\partial d_{31}^\flat}\frac{\partial d_{31}^\flat}{\partial S_1}\frac{S_3}{S_2\sigma_{23}\sqrt{T}} - K\mathrm{e}^{-rT}\frac{\partial \mathcal{N}_3'}{\partial d_{2,1K}}\frac{\partial d_{2,1K}}{\partial S_1}\frac{1}{S_2\sigma_2\sqrt{T}} \\
&= -\left[\mathcal{N}_3''(d_{1K}, d_{12}^\flat, d_{13}^\flat)\left[\frac{1}{S_1\sigma_1\sqrt{T}} + \frac{1}{S_1\sigma_{12}\sqrt{T}} + \frac{1}{S_1\sigma_{13}\sqrt{T}}\right]\frac{S_1}{S_2\sigma_{12}\sqrt{T}}\right. \\
&\quad \left. + \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{1}{S_2\sigma_{12}\sqrt{T}}\right] \\
&\quad - \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}} \\
&\quad - \mathcal{N}_3''(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}}\left[\frac{1}{\sigma_2\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad + \mathcal{N}_3''(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{1}{S_1\sigma_{13}\sqrt{T}}\frac{S_3}{S_2\sigma_{23}\sqrt{T}} \\
&\quad - K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1\sigma_1\sqrt{T}}\frac{1}{S_2\sigma_2\sqrt{T}} \\
&= -\mathcal{N}_3''(d_{1K}, d_{12}^\flat, d_{13}^\flat)\left[\frac{1}{\sigma_1\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right]\frac{1}{S_2\sigma_{12}\sqrt{T}} \\
&\quad - \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat)\frac{1}{S_2\sigma_{12}\sqrt{T}} \\
&\quad - \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}} \\
&\quad - \mathcal{N}_3''(d_{2K}, d_{21}^\flat, d_{23}^\flat)\frac{1}{S_1\sigma_{12}\sqrt{T}}\left[\frac{1}{\sigma_2\sqrt{T}} + \frac{1}{\sigma_{12}\sqrt{T}} + \frac{1}{\sigma_{13}\sqrt{T}}\right] \\
&\quad + \mathcal{N}_3''(d_{3K}, d_{31}^\flat, d_{32}^\flat)\frac{1}{S_1\sigma_{13}\sqrt{T}}\frac{S_3}{S_2\sigma_{23}\sqrt{T}} \\
&\quad - K\mathrm{e}^{-rT}\mathcal{N}_3''(d_{2,1K}, d_{2,2K}, d_{2,3K})\frac{1}{S_1\sigma_1\sqrt{T}}\frac{1}{S_2\sigma_2\sqrt{T}}.
\end{aligned}
$$

The equivalence $\frac{\partial^2 C}{\partial S_1 \partial S_2} = \frac{\partial^2 C}{\partial S_2 \partial S_1}$ confirms our results.

A.3. **Correlation vegas.** Sensitivity of option price to correlation, e.g. ρ_{12}:

$$
\begin{aligned}
\frac{\partial C}{\partial \rho_{12}} &= S_1 \left[\frac{\partial \mathcal{N}_3}{\partial d_{1K}} \frac{\partial d_{1K}}{\partial \rho_{12}} + \frac{\partial \mathcal{N}_3}{\partial d^\flat_{12}} \frac{\partial d^\flat_{12}}{\partial \rho_{12}} \right] + S_2 \left[\frac{\partial \mathcal{N}_3}{\partial d_{2K}} \frac{\partial d_{2K}}{\partial \rho_{12}} + \frac{\partial \mathcal{N}_3}{\partial d^\flat_{21}} \frac{\partial d^\flat_{21}}{\partial \rho_{12}} \right] \\
&\quad -K\mathrm{e}^{-rT} \left[\frac{\partial \mathcal{N}_3}{\partial d_{2,1K}} \frac{\partial d_{2,1K}}{\partial \rho_{12}} + \frac{\partial \mathcal{N}_3}{\partial d_{2,2K}} \frac{\partial d_{2,2K}}{\partial \rho_{12}} \right] \\
&= S_1 \left[\frac{\partial \mathcal{N}_3}{\partial d_{1K}} \frac{\partial d_{1K}}{\partial \sigma_1^2} \frac{\partial \sigma_1^2}{\partial \rho_{12}} + \frac{\partial \mathcal{N}_3}{\partial d^\flat_{12}} \frac{\partial d^\flat_{12}}{\partial \sigma_{12}^2} \frac{\partial \sigma_{12}^2}{\partial \rho_{12}} \right] \\
&\quad +S_2 \left[\frac{\partial \mathcal{N}_3}{\partial d_{2K}} \frac{\partial d_{2K}}{\partial \sigma_2^2} \frac{\partial \sigma_2^2}{\partial \rho_{12}} + \frac{\partial \mathcal{N}_3}{\partial d^\flat_{21}} \frac{\partial d^\flat_{21}}{\partial \sigma_{12}^2} \frac{\partial \sigma_{12}^2}{\partial \rho_{12}} \right] \\
&\quad -K\mathrm{e}^{-rT} \left[\frac{\partial \mathcal{N}_3}{\partial d_{2,1K}} \frac{\partial d_{2,1K}}{\partial \sigma_1^2} \frac{\partial \sigma_1^2}{\partial \rho_{12}} + \frac{\partial \mathcal{N}_3}{\partial d_{2,2K}} \frac{\partial d_{2,2K}}{\partial \sigma_2^2} \frac{\partial \sigma_2^2}{\partial \rho_{12}} \right] \\
&= S_1 \, \mathcal{N}_3'(d_{1K}, d^\flat_{12}, d^\flat_{13}) \left[\frac{\sqrt{T}}{4\sigma_1}(2\sigma_1\sigma_2) + \frac{\sqrt{T}}{4\sigma_{12}}(-2\sigma_1\sigma_2) \right] \\
&\quad +S_2 \, \mathcal{N}_3'(d_{2K}, d^\flat_{21}, d^\flat_{23}) \left[\frac{\sqrt{T}}{4\sigma_2}(2\sigma_1\sigma_2) + \frac{\sqrt{T}}{4\sigma_{12}}(-2\sigma_1\sigma_2) \right] \\
&\quad -K\mathrm{e}^{-rT} \mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K}) \left[\frac{\sqrt{T}}{4\sigma_1}(2\sigma_1\sigma_2) + \frac{\sqrt{T}}{4\sigma_2}(2\sigma_1\sigma_2) \right] \\
&= S_1 \, \mathcal{N}_3'(d_{1K}, d^\flat_{12}, d^\flat_{13}) \frac{\sqrt{T}}{2} \left(\sigma_2 - \frac{\sigma_1\sigma_2}{\sigma_{12}} \right) + S_2 \, \mathcal{N}_3'(d_{2K}, d^\flat_{21}, d^\flat_{23}) \frac{\sqrt{T}}{2} \left(\sigma_1 - \frac{\sigma_1\sigma_2}{\sigma_{12}} \right) \\
&\quad -K\mathrm{e}^{-rT} \mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K}) \frac{\sqrt{T}}{2}(\sigma_1 + \sigma_2),
\end{aligned}
$$

where we made use of

$$\sigma_{12}^2 = \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2.$$

We may generalize this result as follows:

$$
\begin{aligned}
\frac{\partial C}{\partial \rho_{ij}} &= S_i \, \mathcal{N}_3'(d_{iK}, d^\flat_{i\,i+1}, d^\flat_{i\,i+2}) \frac{\sqrt{T}}{2} \left(\sigma_j - \frac{\sigma_i\sigma_j}{\sigma_{ij}} \right) + S_j \, \mathcal{N}_3'(d_{jK}, d^\flat_{j\,j+1}, d^\flat_{j\,j+2}) \frac{\sqrt{T}}{2} \left(\sigma_i - \frac{\sigma_i\sigma_j}{\sigma_{ij}} \right) \\
&\quad -K\mathrm{e}^{-rT} \mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K}) \frac{\sqrt{T}}{2}(\sigma_i + \sigma_j).
\end{aligned}
$$

A.4. **Vegas.** Sensitivity of option price to volatilities, e.g. σ_1:

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma_1} &= S_1 \left[\frac{\partial \mathcal{N}_3}{\partial d_{1K}} \frac{\partial d_{1K}}{\partial \sigma_1^2} \frac{\partial \sigma_1^2}{\partial \sigma_1} + \frac{\partial \mathcal{N}_3}{\partial d_{12}^\flat} \frac{\partial d_{12}^\flat}{\partial \sigma_{12}^2} \frac{\partial \sigma_{12}^2}{\partial \sigma_1} + \frac{\partial \mathcal{N}_3}{\partial d_{13}^\flat} \frac{\partial d_{13}^\flat}{\partial \sigma_{13}^2} \frac{\partial \sigma_{13}^2}{\partial \sigma_1} \right] \\
&\quad + S_2 \frac{\partial \mathcal{N}_3}{\partial d_{21}^\flat} \frac{\partial d_{21}^\flat}{\partial \sigma_{12}^2} \frac{\partial \sigma_{12}^2}{\partial \sigma_1} + S_3 \frac{\partial \mathcal{N}_3}{\partial d_{31}^\flat} \frac{\partial d_{31}^\flat}{\partial \sigma_{13}^2} \frac{\partial \sigma_{13}^2}{\partial \sigma_1} \\
&\quad - K \mathrm{e}^{-rT} \frac{\partial \mathcal{N}_3}{\partial d_{2,1K}} \frac{\partial d_{2,1K}}{\partial \sigma_1^2} \frac{\partial \sigma_1^2}{\partial \sigma_1} \\
&= S_1 \, \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat) \left[\frac{\sqrt{T}}{4\sigma_1} 2\sigma_1 + \frac{\sqrt{T}}{4\sigma_{12}} (2\sigma_1 - 2\rho_{12}\sigma_2) + \frac{\sqrt{T}}{4\sigma_{13}} (2\sigma_1 - 2\rho_{13}\sigma_3) \right] \\
&\quad + S_2 \, \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat) \frac{\sqrt{T}}{4\sigma_{12}} (2\sigma_1 - 2\rho_{12}\sigma_2) + S_3 \, \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat) \frac{\sqrt{T}}{4\sigma_{13}} (2\sigma_1 - 2\rho_{13}\sigma_3) \\
&\quad + K \mathrm{e}^{-rT} \mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K}) \frac{\sqrt{T}}{4\sigma_1} 2\sigma_1 \\
&= S_1 \, \mathcal{N}_3'(d_{1K}, d_{12}^\flat, d_{13}^\flat) \frac{\sqrt{T}}{2} \left(1 + \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{12}} + \frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_{13}} \right) \\
&\quad + S_2 \, \mathcal{N}_3'(d_{2K}, d_{21}^\flat, d_{23}^\flat) \frac{\sqrt{T}}{2} \, \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{12}} + S_3 \, \mathcal{N}_3'(d_{3K}, d_{31}^\flat, d_{32}^\flat) \frac{\sqrt{T}}{2} \, \frac{\sigma_1 - \rho_{13}\sigma_3}{\sigma_{13}} \\
&\quad + K \mathrm{e}^{-rT} \mathcal{N}_3'(d_{2,1K}, d_{2,2K}, d_{2,3K}) \frac{\sqrt{T}}{2}.
\end{aligned}
$$

Similarly for the vegas with respect to σ_2 and σ_3.